Homogeneous Coloring

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joint work with

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Introduction

- Given a vertex-coloring φ, the palette of a vertex v, $P_\varphi(v)$, is the set of colors in the neighborhood of v, i.e.

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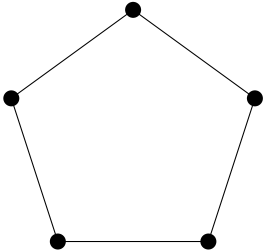
$$P_\varphi(v) = \{\varphi(u) \,|\, u \in N(v)\};$$

- Homogeneous coloring is a proper vertex-coloring such that the pallete sizes of all vertices are the same;
- If $|P_\varphi(v)| = k$ for every $v \rightarrow$ k-homogeneous coloring;

Example: C_5

- 2-homogeneous coloring of C_5:



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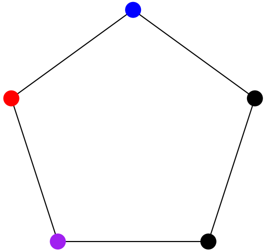

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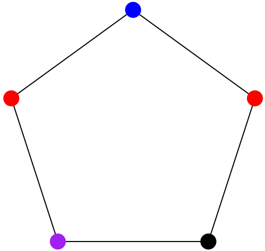

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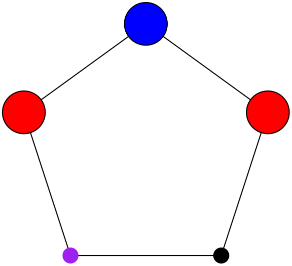

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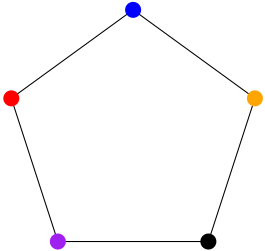

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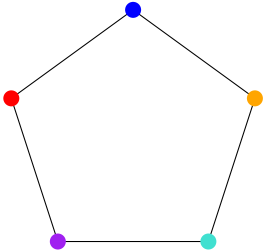

Problems

- Two types of problems:

  (i) Does a graph G admit a k-homogeneous coloring?

  (ii) What is the number of colors needed for a k-homogeneous coloring of G?

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- d-homogeneous coloring admit precisely d-regular graphs;
- $\rightarrow$ only bipartite d-regular graphs can be k-homogeneously colorable for every $k \in \{1, \ldots, d\}$;

- Definition:
  The graphs being k-homogeneously colorable for every $k \in \{1, \ldots, \Delta(G)\}$ are completely homogeneous;

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Conjecture 1 (Šurimová, 2015)

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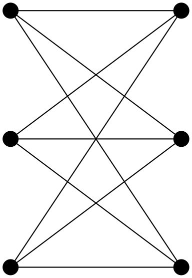

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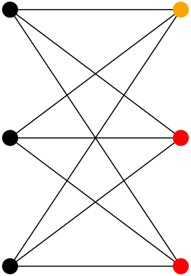

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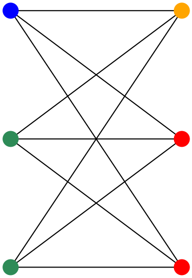

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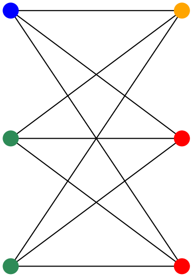

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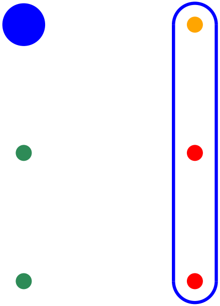

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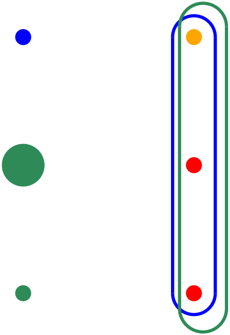

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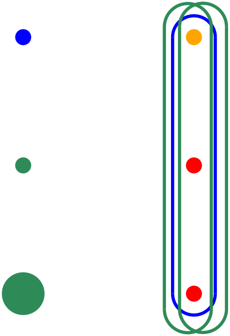

Equivalent Problem

Proposition 2

A graph G admits a k-homogeneous coloring if and only if $\mathcal{H}_G$ admits a vertex-coloring such that in every hyperedge there are vertices of k colors.

- We are interested in bipartite (regular) graphs, so we have two components of the hypergraph: one for each partition of G;

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- Consider each separately;

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Theorem 3 (Henning and Yeo, 2013)

For any integer k, $k \geq 4$, every k-regular k-uniform hypergraph is bipartite.

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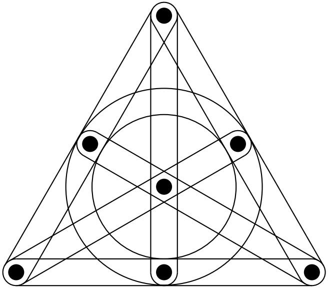

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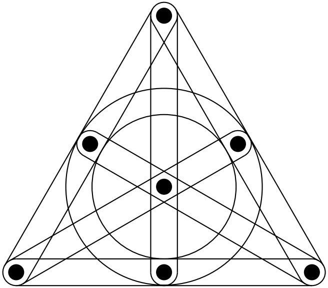

Corollary 4 (Henning and Yeo, 2013)

Every connected 3-regular 3-uniform hypergraph is either bipartite or becomes bipartite after deleting any edge from it.

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Theorem 5 (BL, Madaras, Soták, Šurimová, 2017)

Every 3-regular 3-uniform hypergraph admits an NMNR-coloring with at most 3 colors.

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Corollary 6 (BL, Madaras, Soták, Šurimová, 2017)

Every cubic bipartite graph G admits a 2-homogenous coloring. Moreover,

$$\chi_h^2(G) \leq 6\,.$$

Higher k's

Theorem 7 (Henning and Yeo, 2013)

For any integer $k \geq 4$, every k-regular k-uniform hypergraph is bipartite.

Corollary 8

Every k-regular bipartite graph G, with $k \geq 4$, admits a 2-homogenous coloring. Moreover,

$$\chi_h^2(G) \leq 4\,.$$

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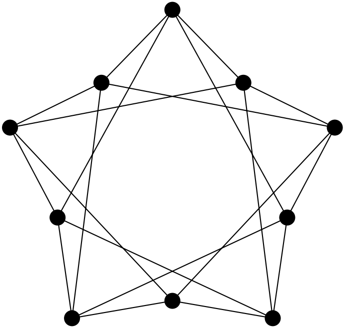
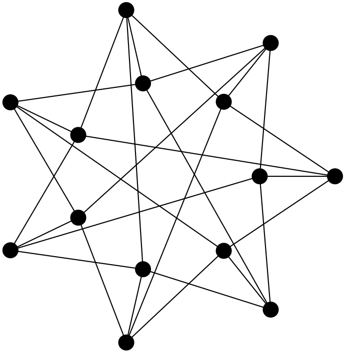
- There exist bipartite 4-regular graphs that are not!
- Graphs below do not admit a 3-homogeneous coloring!





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Thank you!