Repetition Thresholds in Graphs

Borut Lužar

Faculty of Information Studies, Novo mesto, Slovenia

borut.luzar@gmail.com
http://luzar.fis.unm.si

joint work with

Pascal Ochem & Alex Pinlou

Grafy 2019 – June 7, 2019

Basic Notions

- Given an alphabet of k letters

$$\mathbb{A} = \{a_1, a_2, \ldots, a_k\},$$

a word of length n over $\mathbb{A}$,

$$w = \ell_1 \ell_2 \cdots \ell_n,$$

is a sequence of letters from $\mathbb{A}$,
i.e., $\ell_i \in \mathbb{A}$, for every $1 \leq i \leq n$.

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  i.e., $\ell_i \in \mathbb{A}$, for every $1 \leq i \leq n$.
- We are interested in consecutive repetitions of subwords or their parts;

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- A word is non-repetitive if it does not contain a repetition;

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  - from a word w_i we construct a word x_i as a sequence of numbers of zeros between each pair of consecutive ones in w_i;
- Example: $w_6 = 1001011001101001011010011001011 0$

$$x_6 = 210201210120210$$

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  $aba?$
- Thue proved that there is an arbitrarily long word on two letters without three equal consecutive factors;

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- What if we do not require repetition of whole factors?

  $abc\,abc\,ab$ $\quad$ abc is repeated $2 + \frac{2}{3} = \frac{8}{3}$ times;
- How low repetitiveness can be achieved with alphabets on k letters?

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A generalized definition of a repetition:

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$\exp(anana) = \frac{5}{2}$

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- For $k \geq 2$, the repetition threshold $\mathrm{RT}(k)$ for k letters is the smallest α such that there exists an infinite α^{+}-free word over a k-letter alphabet;

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- A word is α^+-free if it contains no β-repetition such that $\beta > \alpha$;
- A word is α-free if it contains no β-repetition such that $\beta \geq \alpha$;
- For $k \geq 2$, the repetition threshold $\mathrm{RT}(k)$ for k letters is the smallest α such that there exists an infinite α^+-free word over a k-letter alphabet;
- The notion of repetition thresholds was initiated by Dejean [11] in 1972;

Repetition Thresholds - Results

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Theorem 1

(*i*) $\mathrm{RT}(2) = 2$ *[11];*
(*ii*) $\mathrm{RT}(3) = \frac{7}{4}$ *[11];*
(*iii*) $\mathrm{RT}(4) = \frac{7}{5}$ *[16];*
(*iv*) $\mathrm{RT}(k) = \frac{k}{k-1}$, *for* $k \geq 5$ *[3, 8, 9, 10, 13, 14, 16, 17].*

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Repetition Thresholds in Graphs

- Study of non-repetitiveness has been generalized to graphs by Currie [4, 5] and Alon et al. [2];
- For a k-vertex coloring of a graph, a sequence of colors on a non-intersecting path is called a factor;
- A vertex coloring is said to be α^+-free (resp. α-free) if every factor is α^+-free (resp. α-free);

Repetition Thresholds in Graphs

- For a k-vertex colored graph G, the repetition threshold is

$$\mathrm{RT}(k, G) = \inf_{k\text{-coloring } c} \sup \{\exp(w) \mid w \text{ is a factor in } c\} .$$

Repetition Thresholds in Graphs

- For a k-vertex colored graph G, the repetition threshold is

$$\mathrm{RT}(k, G) = \inf_{k\text{-coloring } c} \sup \{\exp(w) \mid w \text{ is a factor in } c\} \; .$$

- The repetition threshold over a whole class of graphs $\mathcal{G}$ is defined as

$$\mathrm{RT}(k, \mathcal{G}) = \sup_{G \in \mathcal{G}} \mathrm{RT}(k, G) \, .$$

- For the class of paths $\mathcal{P}$ the repetition thresholds are known, since

$$\mathrm{RT}(k, \mathcal{P}) = \mathrm{RT}(k);$$

Repetition Thresholds of Cycles - $\mathcal{C}$

Theorem 2

(*i*) $\mathrm{RT}(2, \mathcal{C}) = \frac{5}{2}$ *[1];*
(*ii*) $\mathrm{RT}(3, \mathcal{C}) = 2$ *[6];*
(*iii*) $\mathrm{RT}(4, \mathcal{C}) = \frac{3}{2}$ *[7];*
(*iv*) $\mathrm{RT}(5, \mathcal{C}) = \frac{4}{3}$ *[7];*
(*v*) $\mathrm{RT}(k, \mathcal{C}) = 1 + \frac{1}{\lceil \frac{k}{2} \rceil}$*, for* $k \geq 6$ *[12].*

Repetition Thresholds of Trees - $\mathcal{T}$

Theorem 3 ([15])

(i) $\mathrm{RT}(2, \mathcal{T}) = \frac{7}{2}$;
(ii) $\mathrm{RT}(3, \mathcal{T}) = 3$;
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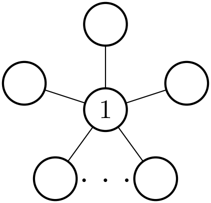

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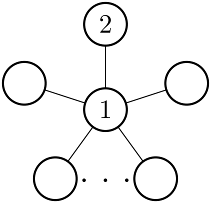

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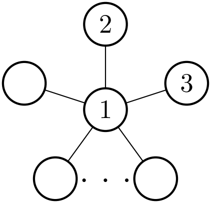

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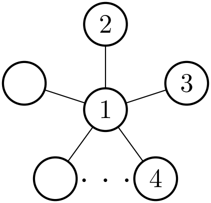

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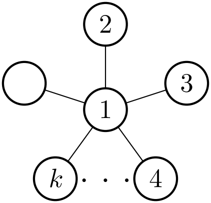

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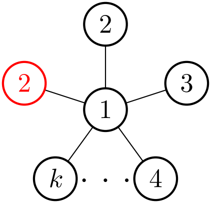

Repetition Thresholds of Subdivisions - $\mathcal{S}$

Theorem 4 ([15])

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- A subdivision of a graph G is a graph obtained from G by a sequence of edge subdivisions. By a graph subdivision, we always mean a "large enough" subdivision.

Repetition Thresholds of Caterpillars - $\mathcal{CP}$

- A caterpillar is a tree such that the graph induced by the vertices of degree at least 2 is a path (we call it a backbone);

Theorem 5 (BL, Ochem, Pinlou, 2018)

(i) $\mathrm{RT}(2, \mathcal{CP}) = 3$;

(ii) $\mathrm{RT}(3, \mathcal{CP}) = 2$;

(iii) $\mathrm{RT}(k, \mathcal{CP}) = \frac{3}{2}$, *for* $k \geq 4$.

RT of Subcubic Caterpillars - $\mathcal{CP}_3$

- Bounding the degree of caterpillars to 3 changes the behavior when the alphabet sizes grow;

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Theorem 6 (BL, Ochem, Pinlou, 2018)

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(*v*) $\mathrm{RT}(k, \mathcal{CP}_3) = 1 + \frac{1}{\lceil \frac{k}{2} \rceil}$, *for* $k \geq 6$.

Proof of $\mathrm{RT}(2, \mathcal{CP}) = \mathrm{RT}(2, \mathcal{CP}_3) = 3$

First, prove $\mathrm{RT}(2, \mathcal{CP}) \leq 3$:

- There is a 2^+-free coloring of backbone

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- There is a 2^+-free coloring of backbone
- Color every leaf with a color distinct from its neighbor

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Second, prove $\mathrm{RT}(2, \mathcal{CP}_3) \geq 3$:

- Two colors $x, y \in \{0, 1\}$

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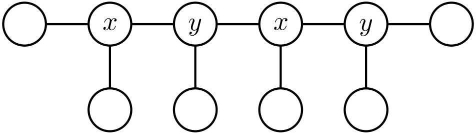

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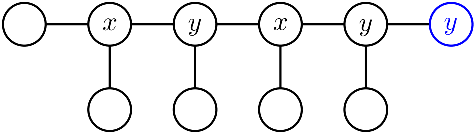

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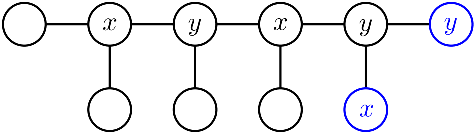

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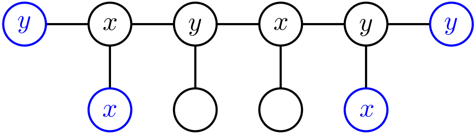

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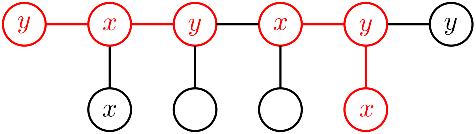

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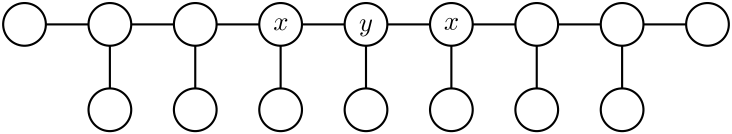

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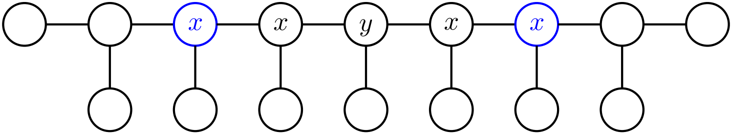

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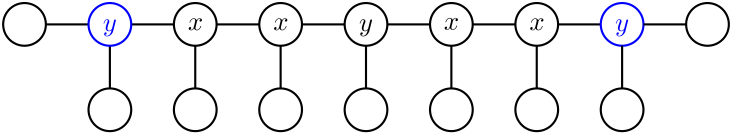

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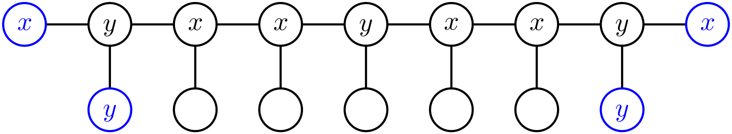

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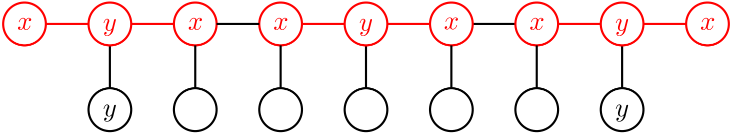
- There is no factor xxx
- $\Rightarrow$ there is no factor $xyxy$ on backbone
- Suppose there is factor xyx on backbone



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- So, the coloring of backbone is comprised of consecutive factors $xxyy$, a contradiction

RT of Subcubic Trees - $\mathcal{T}_3$

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Theorem 7 (BL, Ochem, Pinlou, 2018)

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(*ii*) $\mathrm{RT}(5, \mathcal{T}_3) = \frac{3}{2}$;

(*iii*) $\mathrm{RT}(k, \mathcal{T}_3) = 1 + \frac{1}{2 \log k} + o\left(\frac{1}{\log k}\right)$, *for* $k \geq 6$.

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- Open: $k = 2$ (known: $3 \leq \mathrm{RT}(2, \mathcal{T}_3) \leq \frac{7}{2}$)
- Open: $k = 3$ (known: $2 \leq \mathrm{RT}(3, \mathcal{T}_3) \leq 3$)

Summary

	$\lvert\mathbb{A}\rvert = 2$	$\lvert\mathbb{A}\rvert = 3$	$\lvert\mathbb{A}\rvert = 4$	$\lvert\mathbb{A}\rvert = 5$	$\lvert\mathbb{A}\rvert = k,\ k \geq 6$
$\mathcal{P}$	2	$\frac{7}{4}$	$\frac{7}{5}$	$\frac{5}{4}$	$\frac{k}{k-1}$
$\mathcal{C}$	$\frac{5}{2}$	2	$\frac{3}{2}$	$\frac{4}{3}$	$1 + \frac{1}{\lceil \frac{k}{2} \rceil}$
$\mathcal{S}$	$\frac{7}{3}$	$\frac{7}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
$\mathcal{CP}_3$	3	2	$\frac{3}{2}$	$\frac{4}{3}$	$1 + \frac{1}{\lceil \frac{k}{2} \rceil}$
$\mathcal{T}_3$	?	?	$\frac{3}{2}$	$\frac{3}{2}$	$1 + \frac{1}{2 \log k} + o\left(\frac{1}{\log k}\right)$
$\mathcal{CP}$	3	2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
$\mathcal{T}$	$\frac{7}{2}$	3	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$

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Thank you!