Open problems in S-packing edge-coloring

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Joint work with

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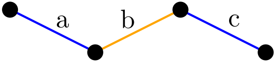
Workshop on Embedded Graphs - Colorings and Structure

June, 2021

Warming-up definitions

- **Subcubic graphs** $\rightarrow$ graphs with maximum degree 3;
- **Simple** graphs (although multiedges are not problematic);
- **Distance between edges** $\rightarrow$ distance between the corresponding vertices in the line graph (adjacent edges are at distance 1);



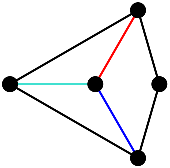
$$d(a, b) = 1,\ d(a, c) = 2$$

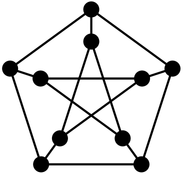
- **Note**: for consistency, the terminology is adjusted in results;

Proper edge-coloring

- **Adjacent edges** receive distinct colors;
- Edges of every color form a **matching**;
- The smallest k for which a graph G admits an edge-coloring with k colors is the **chromatic index** of G, $\chi'(G)$;
- By Vizing's theorem [23], for every subcubic graph G it holds

$$3 \leq \chi'(G) \leq 4$$

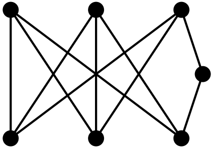


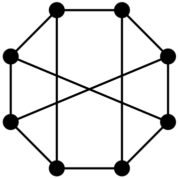


Strong edge-coloring

- **Edges at distance at most** 2 receive distinct colors;
- Edges of every color form an **induced matching**, (i.e., the graph induced on the endvertices is a matching);
- The smallest k for which G admits a strong edge-coloring with k colors is the **strong chromatic index** of G, $\chi'_s(G)$;
- Andersen (1992) [2], and Horák, Qing & Trotter (1993) [10] proved that for every subcubic graph G it holds

$$\chi'_s(G) \leq 10$$



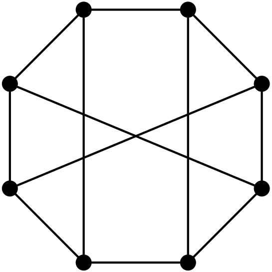


Packings

- A set of edges is a k-**packing** if every pair of edges is at distance at least $k + 1$;
- Hence, every matching is 1-packing and every induced matching is a 2-packing;
- For a non-decreasing sequence of positive integers, $S = (s_1, \dots, s_\ell)$, Gastineau and Togni (2019) [6], defined an S-**packing edge-coloring** of G as a partition of the edge set of G into ℓ subsets $\{X_1, \dots, X_\ell\}$ such that each X_i is an s_i-packing;
- The notion of packing colorings is derived from its vertex analogues introduced by Goddard et al. (2008) [7], and Goddard and Xu (2012) [8];

Example

- Consider the cubic graph with $\chi'_s = 10$:



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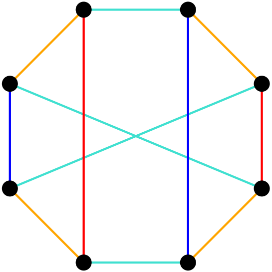

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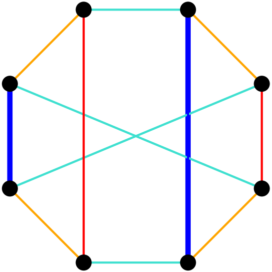

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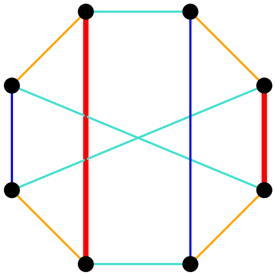

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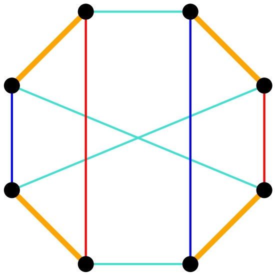

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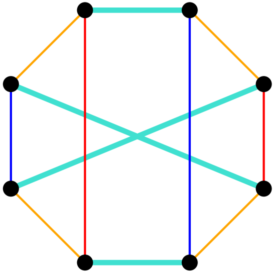

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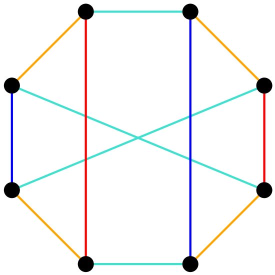

- It is $(1, 1, 2, 2)$-packing edge-colorable;
- What is the smallest k so it is $(1, \underbrace{2, ..., 2}_{k})$-packing edge-colorable?

Today's focus

- For $S = (s_1, \ldots, s_\ell)$, we are interested in S-packing edge-colorings with $s_i \in \{1, 2\}$ for a given number of 1's;
- We abbreviate $(\underbrace{1, \ldots, 1}_{p}, \underbrace{2, \ldots, 2}_{q}) = (1^p, 2^q)$;
- By Vizing (also Brooks):
  Every subcubic graph admits a (1^4)-packing edge-coloring.
- By Andersen and Horák et al.:
  Every subcubic graph admits a (2^{10})-packing edge-coloring.
- **What is in between** and **what is open**?

$$(1, 1, 1, 2^k)$$

$(1,1,1,2)$-packing edge-coloring

- The color appearing the least number of times, call it δ, in a proper edge-coloring of subcubic graphs is rare;
- Albertson & Haas (1996) [1]:
  If G is cubic, at most $\frac{2}{15}$ edges are colored with δ.
- Steffen (2004) [20]:
  The Petersen graph is the only bridgeless cubic graph achieving $\frac{2}{15}$ edges colored with δ.
- Fouquet & Vanherpe (2013) [5] and
  (also for multigraphs) Kamiński & Kowalik (2014)[11]:
  For any subcubic graph, less than $\frac{2}{15}$ edges are colored with δ, except for three graphs.
- As a side product...

$(1, 1, 1, 2)$-packing edge-coloring

- Fouquet & Vanherpe (2013) [5] and Payan (1977) [18]: Every subcubic graph admits a $(1, 1, 1, 2)$-packing edge-coloring.
- Here a 2-packing cannot be replaced by a 3-packing due to the Petersen and the Tietze graphs.

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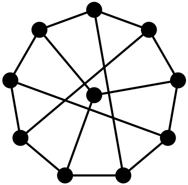
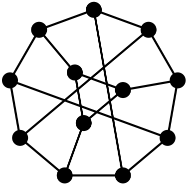
- Here a 2-packing cannot be replaced by a 3-packing due to the Petersen and the Tietze graphs.

Conjecture 1 (Gastineau & Togni [6])

Every cubic graph different from the Petersen and the Tietze graph is (1, 1, 1, 3)*-packing edge-colorable.*

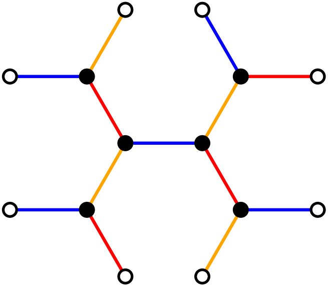




$(1, 1, 2^k)$

$(1, 1, 2^k)$-packing edge-coloring

- Trivial: $k \leq 6$
  Take any $(1, 1, 1, 2)$-packing edge-coloring of G and replace one 1-packing with five 2-packings;



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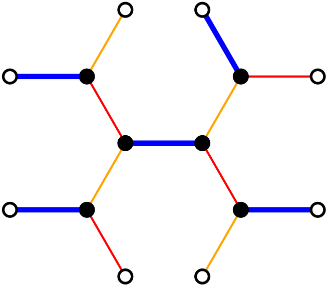

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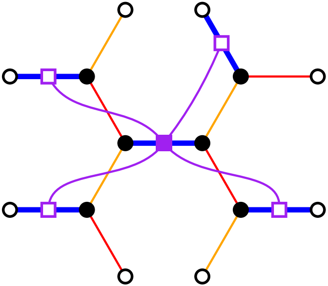
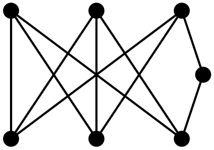


$(1, 1, 2^k)$-packing edge-coloring

- Showing there is a 1-packing A in a $(1, 1, 1, 2)$-packing edge-coloring such that no five edges of A are pairwise at distance 2, Gastineau & Togni (2019) [6] proved: Every bridgeless cubic graph admits a $(1, 1, 2^5)$-packing edge-coloring.
- Hocquard, Lajou & BL (2020^+) [9]: Every subcubic graph admits a $(1, 1, 2^5)$-packing edge-coloring.

$(1, 1, 2^k)$-packing edge-coloring

- There are graph(s) that do not admit a $(1, 1, 2^3)$-packing edge-coloring;



Conjecture 2 (Gastineau and Togni [6])

Every subcubic graph is $(1, 1, 2^4)$-packing edge-colorable.

- The conjecture is supported by checking all bridgeless subcubic graphs on at most 17 vertices;
- Is $K_{3,3}$ with a subdivided edge the only bridgeless subcubic graph needing four 2-packings?

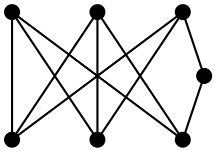
$(1, 2^k)$

$(1, 2^k)$-packing edge-coloring

- Trivial: $k \leq 9$
  By the strong edge-coloring result;
- Hocquard, Lajou & BL (2020^+) [9]:
  Every subcubic graph admits a $(1, 2^8)$-packing edge-coloring.
- (Using the fact that the 1-packing contains many edges, the proof is rather simple.)

$(1, 2^k)$-packing edge-coloring

- There are graph(s) that do not admit a $(1, 2^6)$-packing edge-coloring;



- Gastineau and Togni [6] asked, but we

Conjecture 3 (Hocquard, Lajou & BL (2020^+) [9])

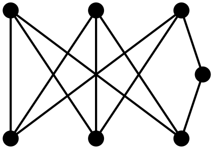
Every subcubic graph is $(1, 2^7)$-packing edge-colorable.

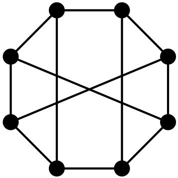
- The conjecture is supported by checking all bridgeless subcubic graphs on at most 17 vertices;
- Is $K_{3,3}$ with a subdivided edge the only bridgeless subcubic graph needing seven 2-packings?

(2^k)

(2^k)-packing edge-coloring

- A.k.a. strong edge-coloring;
- We know that $k = 10$, but...
- We only know two bridgeless subcubic graphs achieving the bound:





- Are they the only ones?

Why interesting?

How they are connected?

- The (conjectured) bounds series:

$$(1,1,1,2) \qquad (1,1,2^4) \qquad (1,2^7) \qquad (2^{10})$$

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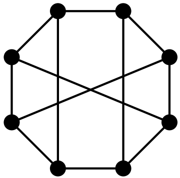
- It **does not** apply to class I graphs!



Adding constraints

The Conjecture

Conjecture 4 (Faudree, Gyárfás, Schelp & Tuza (1990) [4])

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(4) *If G is bipartite and for each edge uv we have $d(u) + d(v) \leq 5$, then it admits a (2^{6})-packing edge-coloring;*

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(5) *If G is bipartite of girth at least 6, then it admits a (2^{7})-packing edge-coloring;*

(6) *If G is bipartite and has girth large enough, then it admits a (2^{5})-packing edge-coloring;*

Adding constraints: Class I graphs

Class I graphs

- By definition:
  Subcubic class I graphs are $(1,1,1)$-packing edge-colorable;

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Class I graphs

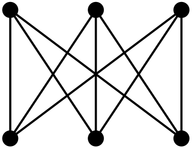
Conjecture 5 (Gastineau and Togni [6])

Every subcubic class I graph is $(1, 1, 2^3)$-packing edge-colorable.

Conjecture 6 (Hocquard, Lajou & BL (2020^+) [9])

Every subcubic class I graph is $(1, 2^6)$-packing edge-colorable.

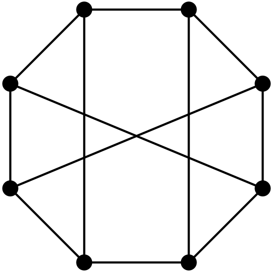
- Both conjectures, if true, are tight, due to $K_{3,3}$;



Class I graphs

Question 7

Is the Wagner graph the only subcubic class I graph that does not admit a (2^9)-packing edge-coloring?



Adding constraints:
Planar graphs

Planar graphs

- By Tait [22] and the Four Color Theorem:
  Every bridgeless cubic planar graph admits a $(1, 1, 1)$-packing edge-coloring.
- Must be cubic and bridgeless $\rightarrow$ K_4 with a subdivided edge is not class I.

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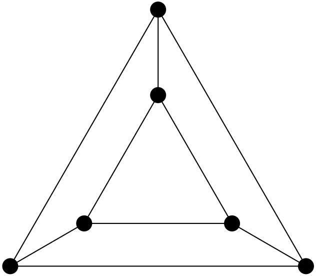
Conjecture 8 (Albertson & Haas (1996) [1])

Every bridgeless subcubic planar graph with at least two vertices of degree 2 admits a $(1,1,1)$-packing edge-coloring.

- Special case of Seymour's conjecture [19];
- Only partially solved;

Planar graphs

- Kostochka et al. (2016) [14]:
  Every subcubic planar graph admits a (2^9)-packing edge-coloring.
- The bound is tight due to the 3-prism, which is the only known subcubic planar graph that does not admit a (2^8)-packing edge-coloring.



Planar graphs

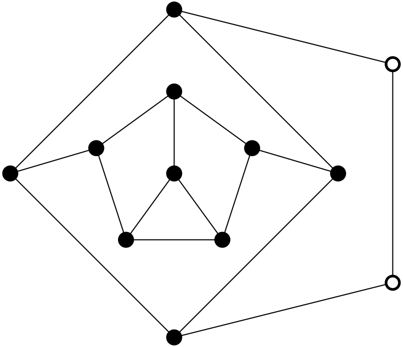
- No particular bounds for general subcubic planar graphs in terms of $(1, 1, 2^k)$- and $(1, 2^\ell)$-packing edge-coloring;

Conjecture 9 (Hocquard, Lajou & BL (2020^+) [9])

Every subcubic planar graph is $(1, 1, 2^3)$*-packing edge-colorable and* $(1, 2^6)$*-packing edge-colorable.*

Planar graphs

- Infinitely many subcubic bridgeless planar graphs which do not admit a $(1, 2^5)$-packing edge-coloring;
- Infinitely many subcubic bridgeless planar graphs which do not admit a $(1, 1, 2^2)$-packing edge-coloring;



Adding constraints: Bipartite graphs

Bipartite graphs

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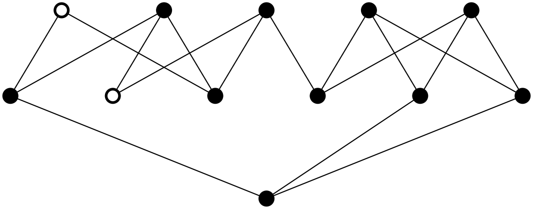
- By Steger & Yu (1993) [21]:
  Every subcubic bipartite graph admits a (2^9)-packing edge-coloring.
- Known bridgeless graphs that do not admit (2^8)-packing edge-coloring have less than 13 vertices;

Conjecture 10 (BL, Mačajová, Škoviera & Soták (2021^+) [15])

If G is a bridgeless [bipartite] subcubic graph on at least 13 vertices, then it is (2^8)-packing edge-colorable.

Bipartite graphs

- Bipartite graphs are class I, so the results apply;
- Conjectures 5 and 6 can be considered in a special case of bipartite graphs;
- Even in the bipartite setting there are many bridgeless graphs achieving the upper bound;



Bipartite graphs - Light edges

- Recall item (4) of Conjecture 12:

Conjecture 11 (Faudree, Gyárfás, Schelp & Tuza (1990) [4])

For every subcubic graph G it holds:

(4) *If G is bipartite and for each edge uv it holds $w(uv) \leq 5$, then it admits a (2^6)-packing edge-coloring*;

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Conjecture 11 (Faudree, Gyárfás, Schelp & Tuza (1990) [4])

For every subcubic graph G it holds:

(4) *If G is bipartite and for each edge uv it holds $w(uv) \leq 5$, then it admits a (2^6)-packing edge-coloring;*

- "Equivalent" to the problem of incidence coloring of subcubic graphs, solved by Maydanskiy [17] in 2005;
- Solved also by Wu and Lin [24] in 2008...
- ... and also by BL, Mockovčiaková, Soták & Škrekovski [16] in 2013;

Bipartite graphs - Light edges

- Bipartite graphs with edges of weight at most 5 need 3 colors for a proper edge-coloring;
- Is it true that every subcubic bipartite graph with each edge of weight 5 admits a $(1, 1, 2^2)$-packing edge-coloring?

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- Is it true that every subcubic bipartite graph with each edge of weight 5 admits a $(1, 2^4)$-packing edge-coloring?
- Both bounds are tight by the lower bounds for the trees...

Adding constraints:
Big girth graphs

Increasing girth - Trees

- Every tree admits:
  (1) a $(1,1,1)$-packing edge-coloring;
  (2) a $(1,1,2^2)$-packing edge-coloring;
  (3) a $(1,2^4)$-packing edge-coloring;
  (4) a (2^5)-packing edge-coloring;
- The bounds are tight (consider a neighborhood of one edge);

Big girth

- Is it possible for subcubic graphs of large enough girth to have the same bounds as trees?

Big girth

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- Not for proper edge-coloring!

Big girth

- Is it possible for subcubic graphs of large enough girth to have the same bounds as trees?
- Not for proper edge-coloring!
- Kochol (1996) [12]:
  There are snarks of arbitrary large girth,
  i.e., there are bridgeless cubic graphs with arbitrary large girth that do not admit a $(1, 1, 1)$-packing edge-coloring.

Big girth

Conjecture 12 (Faudree, Gyárfás, Schelp & Tuza (1990) [4])

For every subcubic graph G it holds:

(5) *If G is bipartite of girth at least 6, then it admits a (2^7)-packing edge-coloring;*

(6) *If G is bipartite and has girth large enough, then it admits a (2^5)-packing edge-coloring;*

- Item (5) of Conjecture 12 is open;
- Item (6) of Conjecture 12 has been rejected;
- BL, Mačajová, Škoviera & Soták (2021^+) [15]
  A cubic graph G admits a (2^5)-packing edge-coloring if and only if G covers the Petersen graph.

Big girth

- Some partial results for (2^5)-packing edge-coloring:
  De Orsey et al. (2018) [3]:
  Every subcubic planar graph of girth at least 30 admits a (2^5)-packing edge-coloring.
- Analogues of item (6) are the $(1, 1, 2^2)$-packing and the $(1, 2^4)$-packing edge-coloring;
- **But!** Gastineau & Togni (2019) [6]:
  Every cubic graph admitting a $(1, 1, 2^2)$-packing edge-coloring is class 1 and has order divisible by four!
- So, it does not hold for $(1, 1, 2^2)$-packing edge-coloring;

Big girth

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- **But!** Gastineau & Togni (2019) [6]:
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- So, it does not hold for $(1,1,2^2)$-packing edge-coloring;
- Is it true that every bipartite subcubic graph of large enough girth admits a $(1,2^4)$-packing edge-coloring?

Summary

	$(1^3, 2^a)$ (a_L, a_U)	$(1^2, 2^b)$ (b_L, b_U)	$(1, 2^c)$ (c_L, c_U)	(2^d) (d_L, d_U)
general	[1,1]	[4,5]	[7,8]	[10,10]
class I	[0,0]	[3,4]	[6,7]	[10,10]
planar	[1,1]	[3,5]	[6,8]	[9,9]
bipartite	[0,0]	[3,4]	[6,7]	[9,9]
large girth	[1,1]	[3,5]	[4,8]	[6,10]

- $x \in \{a, b, c, d\}$;
- x_L - exists graph which needs at least so many 2-packings;
- x_U - proven upper bound for the number of 2-packings;
- red color - only finitely many known examples of bridgeless graphs attaining the bound;
- blue color - our result;
- green color - resolved completely;

Thank you!

References

I

[1] ALBERTSON, M. O., AND HAAS, R.
Parsimonious edge coloring.
Discrete Math. 148 (1996), 1–7.

[2] ANDERSEN, L. D.
The strong chromatic index of a cubic graph is at most 10.
Discrete Math. 108 (1992), 231–252.

[3] DEORSEY, P., FERRARA, M., GRABER, N., HARTKE, S., NELSEN, L., SULLIVAN, E., JAHANBEKAM, S., LIDICKÝ, B., STOLEE, D., AND WHITE, J.
On the strong chromatic index of sparse graphs.
Electron. J. Combin. 25, 3 (2018).

References II

[4] Faudree, R. J., Gyárfás, A., Schelp, R. H., and Tuza, Z.
The strong chromatic index of graphs.
Ars Combin. 29B (1990), 205–211.

[5] Fouquet, J.-L., and Vanherpe, J.-M.
On Parsimonious Edge-Colouring of Graphs with Maximum Degree Three.
Graphs Combin. 29, 3 (2013), 475–487.

[6] Gastineau, N., and Togni, O.
On S-packing edge-colorings of cubic graphs.
Discrete Appl. Math. 259 (2019), 63–75.

References

III

[7] GODDARD, W., HEDETNIEMI, S. M., HEDETNIEMI, S. T., HARRIS, J. M., AND RALL, D. F.
Broadcast chromatic numbers of graphs.
Ars Combin. 86 (2008), 33–49.

[8] GODDARD, W., AND XU, H.
The S-packing chromatic number of a graph.
Discuss. Math. Graph Theory 32 (2012), 795–806.

[9] HOCQUARD, H., LAJOU, D., AND LUŽAR, B.
Between proper and strong edge-colorings of subcubic graphs.
ArXiv Preprint (2020).
http://arxiv.org/abs/2011.02175.

References IV

[10] Horák, P., Qing, H., and Trotter, W. T.
Induced matchings in cubic graphs.
J. Graph Theory 17, 2 (1993), 151–160.

[11] Kamiński, M., and Kowalik, L.
Beyond the vizing's bound for at most seven colors.
SIAM Journal on Discrete Mathematics 28, 3 (2014), 1334–1362.

[12] Kochol, M.
Snarks without Small Cycles.
J. Combin. Theory Ser. B 67, 1 (1996), 34–47.

References

V

[13] König, D.
Über Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre.
Math. Ann. 77 (1916), 453–465.

[14] Kostochka, A., Li, X., Ruksasakchai, W., Santana, M., Wang, T., and Yu, G.
Strong chromatic index of subcubic planar multigraphs.
Europ. J. Combin. 51 (2016), 380–397.

[15] Lužar, B., Mačajová, M., Škoviera, M., and Soták, R.
Strong edge colorings of graphs and the covers of Kneser graphs.
ArXiv Preprint (2021).
http://arxiv.org/abs/2101.04768.

References VI

[16] Lužar, B., Mockovčiaková, M., Soták, R., and Škrekovski, R.
Strong edge coloring ofsubcubic bipartite graphs.
ArXiv Preprint (2013).
http://arxiv.org/abs/1311.6668.

[17] Maydanskiy, M.
The incidence coloring conjecture for graphs of maximum degree 3.
Discrete Math. 292, 1–3 (2005), 131–141.

References VII

[18] Payan, C.

Sur quelques problmes de couverture et de couplage en combinatoire.

PhD thesis, Institut National Polytechnique de Grenoble - INPG, Université Joseph-Fourier - Grenoble I, 1977.

In French.

[19] Seymour, P. D.

On Tutte's Extension of the Four-Color Problem.

J. Combin. Theory Ser. B 31 (1981), 82–94.

[20] Steffen, E.

Measurements of edge-uncolorability.

Discrete Math. 280, 1 (2004), 191–214.

VIII

[21] STEGER, A., AND YU, M.-L.
On induced matchings.
Discrete Math. 120 (1993), 291–295.

[22] TAIT, P. G.
On the colouring of maps.
Proc. R. Soc. Edinburgh Sect. A 10 (1880), 501–503, 729.

[23] VIZING, V. G.
On an estimate of the chromatic class of a p-graph.
Metody Diskret. Analiz 3 (1964), 25–30.

[24] WU, J., AND LIN, W.
The strong chromatic index of a class of graphs.
Discrete Math. 308 (2008), 6254–6261.