Repetition Thresholds in Graphs [12]

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joint work with

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Basic Notions

- Given an alphabet

$$\mathbb{A} = \{a_1, a_2, \ldots, a_k\}$$

of k letters, a word

$$w = \ell_1 \ell_2 \cdots \ell_n$$

of length n over $\mathbb{A}$ is a sequence of letters from $\mathbb{A}$, i.e., $\ell_i \in \mathbb{A}$, for every $1 \leq i \leq n$.

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  of length n over $\mathbb{A}$ is a sequence of letters from $\mathbb{A}$, i.e., $\ell_i \in \mathbb{A}$, for every $1 \leq i \leq n$.
- We are interested in consecutive repetitions of subwords or their parts

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- A word is non-repetitive if it contains no repetition

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  - from a word w_i we construct a word x_i as a sequence of numbers of zeros between each pair of consecutive ones in w_i
- Example: $w_6 = 1001011001101001011010011001011 0$

$$x_6 = 210201210120210$$

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- Thue proved that there is an arbitrarily long word on two letters without three equal consecutive factors

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Repetition Thresholds

- What if we do not require repetition of whole factors?

  abcabcab *abc* is repeated $2 + \frac{2}{3} = \frac{8}{3}$ times
- How low repetitiveness can be achieved with alphabets on k letters?

Repetition Thresholds

A generalized definition of a repetition:

- A prefix of a word $w = w_1 \ldots w_r$ is a word $p = w_1 \ldots w_s$, for some $s \leq r$

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$\exp(anana) = \frac{5}{2}$

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- For $k \geq 2$, the repetition threshold $\mathrm{RT}(k)$ for k letters is the smallest α such that there exists an infinite α^+-free word over a k-letter alphabet
- The notion of repetition thresholds was initiated by Dejean [10] in 1972

Repetition Thresholds - Results

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Theorem 1

(*i*) $\mathrm{RT}(2) = 2$ *[10];*

(*ii*) $\mathrm{RT}(3) = \frac{7}{4}$ *[10];*

(*iii*) $\mathrm{RT}(4) = \frac{7}{5}$ *[16];*

(*iv*) $\mathrm{RT}(k) = \frac{k}{k-1}$, *for* $k \geq 5$ *[3, 7, 8, 9, 13, 14, 16, 17].*

Repetition Thresholds in Graphs

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Repetition Thresholds in Graphs

- Study of non-repetitiveness has been generalized to graphs by Currie [4, 5] and Alon et al. [2]
- For a k-vertex coloring of a graph, a sequence of colors on a non-intersecting path is called a factor
- A vertex coloring is said to be α^+-free (resp. α-free) if every factor is α^+-free (resp. α-free)

Repetition Thresholds in Graphs

- For a k-vertex colored graph G, the repetition threshold is

$$\mathrm{RT}(k, G) = \inf_{k\text{-coloring } c} \sup \{\exp(w) \mid w \text{ is a factor in } c\} .$$

Repetition Thresholds in Graphs

- For a k-vertex colored graph G, the repetition threshold is

$$\mathrm{RT}(k,G) = \inf_{k\text{-coloring } c} \sup \{\exp(w) \mid w \text{ is a factor in } c\} .$$

- The repetition threshold over a whole class of graphs $\mathcal{G}$ is defined as

$$\mathrm{RT}(k,\mathcal{G}) = \sup_{G \in \mathcal{G}} \mathrm{RT}(k,G) .$$

- For the class of paths $\mathcal{P}$ the repetition thresholds are known, since

$$\mathrm{RT}(k,\mathcal{P}) = \mathrm{RT}(k)$$

Repetition Thresholds of Cycles - $\mathcal{C}$

Theorem 2

(*i*) $\mathrm{RT}(2, \mathcal{C}) = \frac{5}{2}$ *[1];*

(*ii*) $\mathrm{RT}(3, \mathcal{C}) = 2$ *[6];*

(*iii*) $\mathrm{RT}(k, \mathcal{C}) = 1 + \frac{1}{\lceil \frac{k}{2} \rceil}$, *for* $k \geq 6$ *[11].*

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Conjecture 3 ([11])

(*i*) $\mathrm{RT}(4, \mathcal{C}) = \frac{3}{2}$;

(*ii*) $\mathrm{RT}(5, \mathcal{C}) = \frac{4}{3}$.

Repetition Thresholds of Trees - $\mathcal{T}$

Theorem 4 ([15])

(*i*) $\mathrm{RT}(2, \mathcal{T}) = \frac{7}{2}$;
(*ii*) $\mathrm{RT}(3, \mathcal{T}) = 3$;
(*iii*) $\mathrm{RT}(k, \mathcal{T}) = \frac{3}{2}$, *for* $k \geq 4$.

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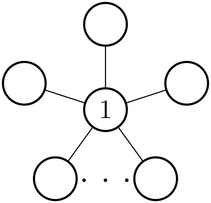

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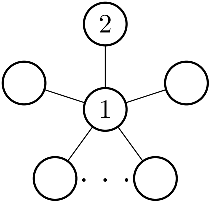

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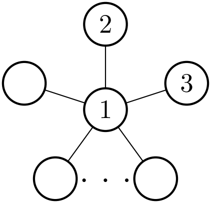

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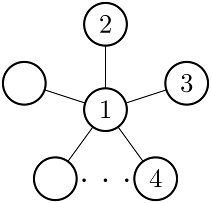

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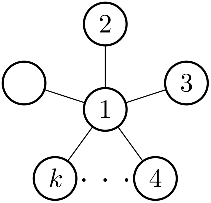

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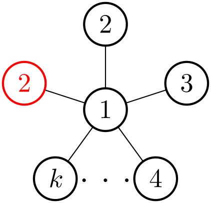

Repetition Thresholds of Subdivisions - $\mathcal{S}$

Theorem 5 ([15])

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(ii) $\mathrm{RT}(3, \mathcal{S}) = \frac{7}{4}$*;*
(iii) $\mathrm{RT}(k, \mathcal{S}) = \frac{3}{2}$*, for* $k \geq 4$*.*

- A subdivision of a graph G is a graph obtained from G by a sequence of edge subdivisions. By a graph subdivision, we always mean a “large enough” subdivision

Repetition Thresholds of Caterpillars - $\mathcal{CP}$

- A caterpillar is a tree such that the graph induced by the vertices of degree at least 2 is a path (we call it a backbone)

Theorem 6 (BL, Ochem, Pinlou, 2018^{+})

(*i*) $\mathrm{RT}(2, \mathcal{CP}) = 3$;

(*ii*) $\mathrm{RT}(3, \mathcal{CP}) = 2$;

(*iii*) $\mathrm{RT}(k, \mathcal{CP}) = \frac{3}{2}$, *for* $k \geq 4$.

RT of Subcubic Caterpillars - $\mathcal{CP}_3$

- Bounding the degree of caterpillars to 3 changes the behavior when the alphabet sizes grow

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Theorem 7 (BL, Ochem, Pinlou, 2018$^+$)

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(v) $\mathrm{RT}(k, \mathcal{CP}_3) = 1 + \frac{1}{\lceil \frac{k}{2} \rceil}$, for $k \geq 6$.

Proof of $\mathrm{RT}(2, \mathcal{CP}) = \mathrm{RT}(2, \mathcal{CP}_3) = 3$

First, prove $\mathrm{RT}(2, \mathcal{CP}) \leq 3$:

- There is a 2^+-free coloring of backbone

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- There is a 2^+-free coloring of backbone
- Color every leaf with a color distinct from its neighbor

Proof of $\mathrm{RT}(2, \mathcal{CP}) = \mathrm{RT}(2, \mathcal{CP}_3) = 3$

Second, prove $\mathrm{RT}(2, \mathcal{CP}_3) \geq 3$:

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- Suppose there is factor *xyxy* on backbone

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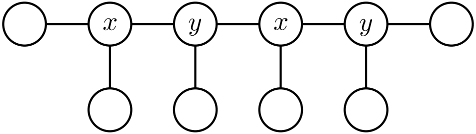

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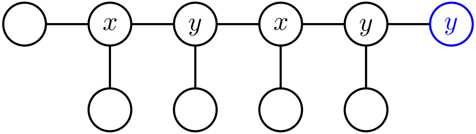

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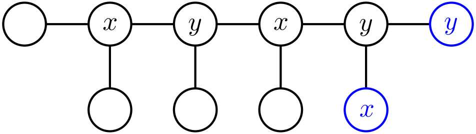

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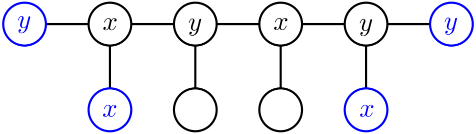

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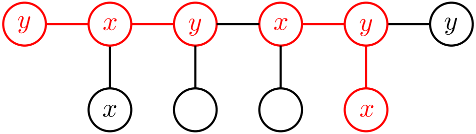

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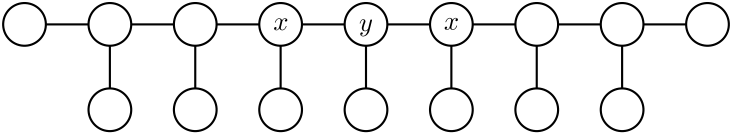

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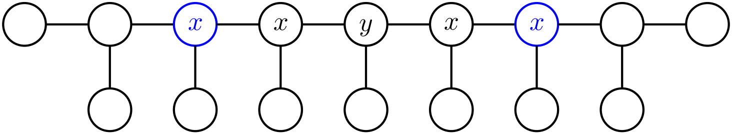

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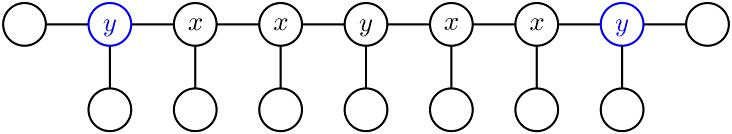

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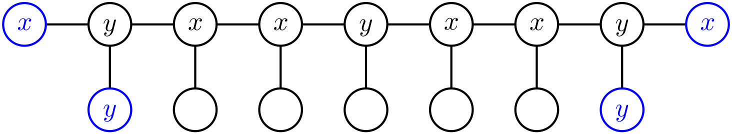

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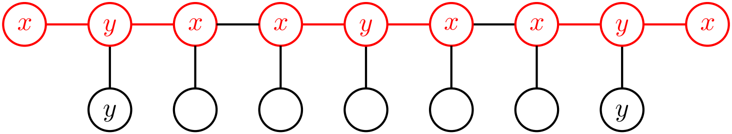

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RT of Subcubic Trees - $\mathcal{T}_3$

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(*iii*) $\mathrm{RT}(k, \mathcal{T}_3) = 1 + \frac{1}{2\log k} + o\left(\frac{1}{\log k}\right)$, *for* $k \geq 6$.

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- Open: $k = 2$ (known: $3 \leq \mathrm{RT}(2, \mathcal{T}_3) \leq \frac{7}{2}$)
- Open: $k = 3$ (known: $2 \leq \mathrm{RT}(3, \mathcal{T}_3) \leq 3$)

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- We prove it in a stronger form:
  $\mathrm{RT}((t+1)2^{\lfloor (t+1)/2 \rfloor}, \mathcal{T}_3) \leq 1 + \frac{1}{t} \leq \mathrm{RT}(3(2^{\lfloor t/2 \rfloor} - 1), \mathcal{T}_3), t \geq 4$

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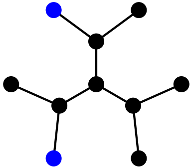

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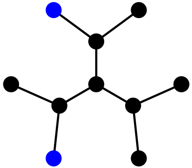

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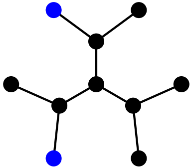

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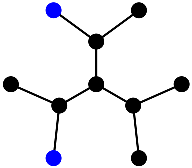

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  - since they have the same λ, their common ancestor is at distance at least $\lfloor (t-1)/2 \rfloor + 1$ from each of them, so they are at distance at least $2(\lfloor (t-1)/2 \rfloor + 1) \geq t$, a contradiction.

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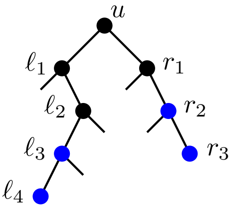

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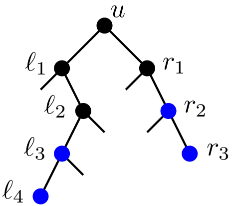

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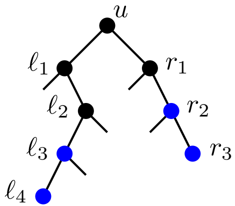

- Since ℓ_{k-1} is the parent of ℓ_k and r_{j-1} is the parent of r_j, the γ-components cannot match
- So, all the vertices of p are on different levels, which means w' and consequently w are not $(1 + \frac{1}{t})^+$-free, a contradiction

Summary

	$\lvert\mathbb{A}\rvert = 2$	$\lvert\mathbb{A}\rvert = 3$	$\lvert\mathbb{A}\rvert = 4$	$\lvert\mathbb{A}\rvert = 5$	$\lvert\mathbb{A}\rvert = k,\ k \geq 6$
$\mathcal{P}$	2	$\frac{7}{4}$	$\frac{7}{5}$	$\frac{5}{4}$	$\frac{k}{k-1}$
$\mathcal{C}$	$\frac{5}{2}$	2	?	?	$1 + \frac{1}{\lceil \frac{k}{2} \rceil}$
$\mathcal{S}$	$\frac{7}{3}$	$\frac{7}{4}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
$\mathcal{CP}_3$	3	2	$\frac{3}{2}$	$\frac{4}{3}$	$1 + \frac{1}{\lceil \frac{k}{2} \rceil}$
$\mathcal{T}_3$	?	?	$\frac{3}{2}$	$\frac{3}{2}$	$1 + \frac{1}{2\log k} + o\left(\frac{1}{\log k}\right)$
$\mathcal{CP}$	3	2	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
$\mathcal{T}$	$\frac{7}{2}$	3	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$

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Thank you!