Some coloring properties of medial graphs

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joint work with
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KoKoS

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The Problem

Problem 1 (Czap, Jendroľ & Voigt [3])

Is there a bipartite plane graph such that its medial graph has chromatic number 4?

In other words:

Is there a bipartite plane graph that needs 4 colors for facially-proper edge-coloring?

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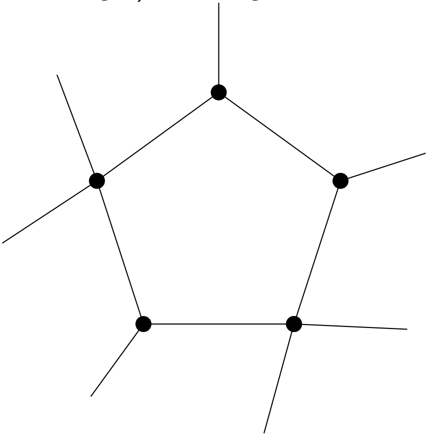

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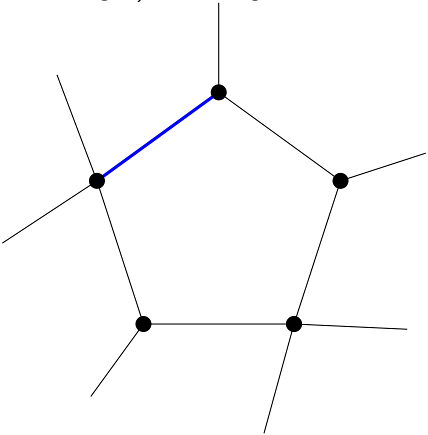

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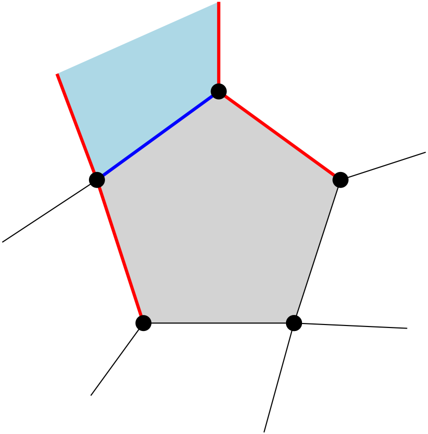

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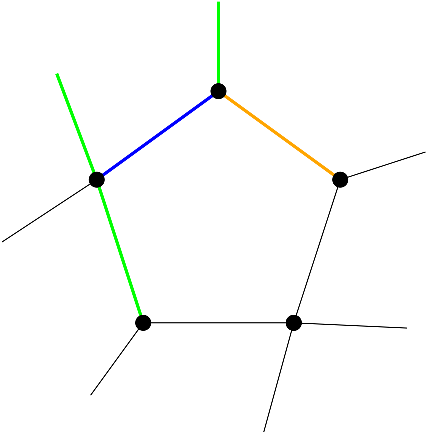

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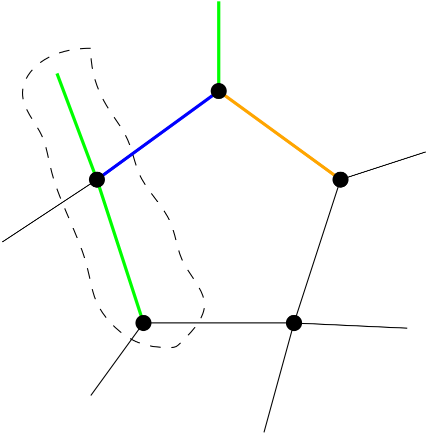

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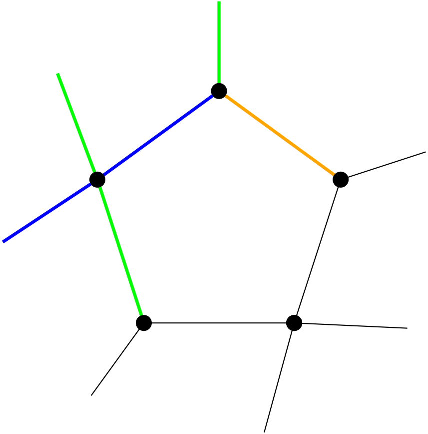

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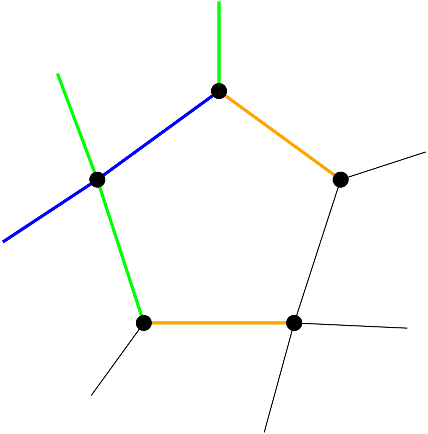

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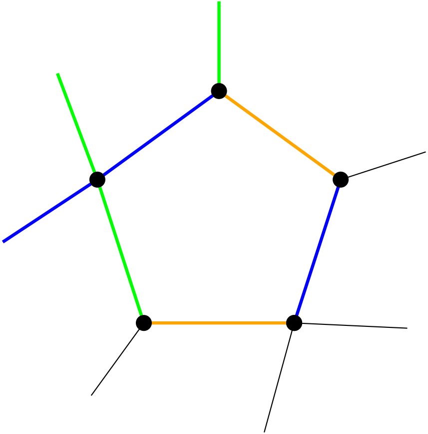

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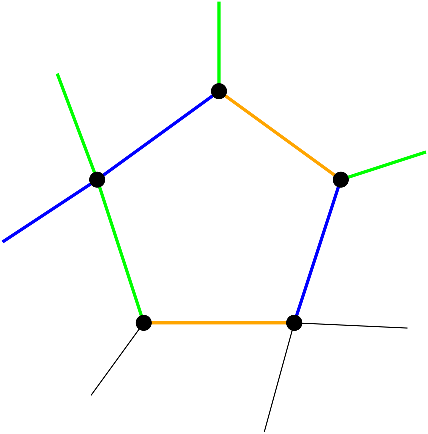

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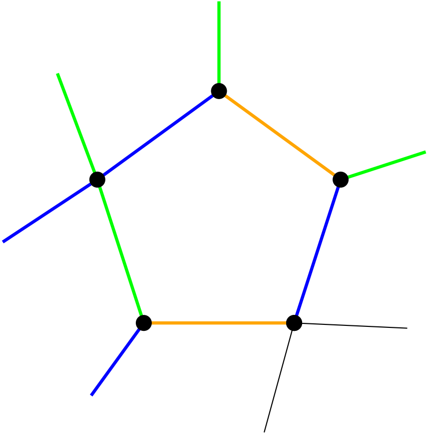

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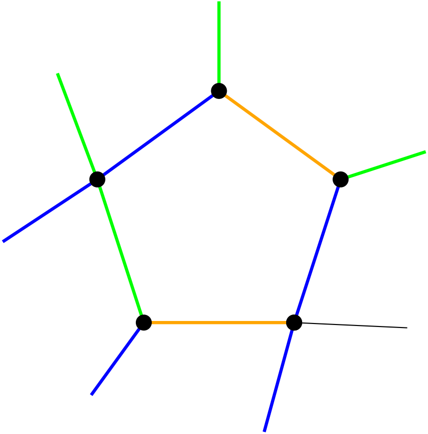

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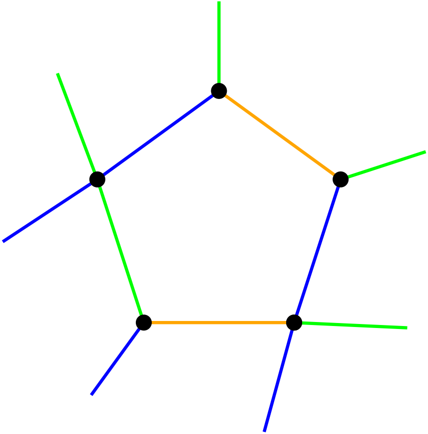

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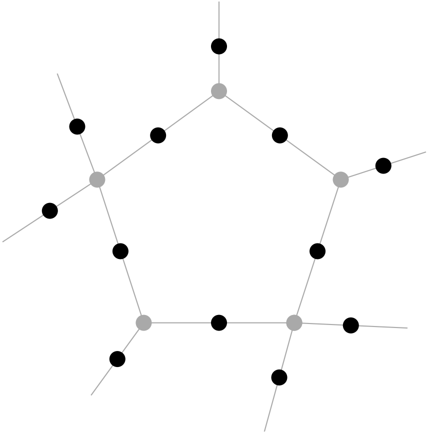

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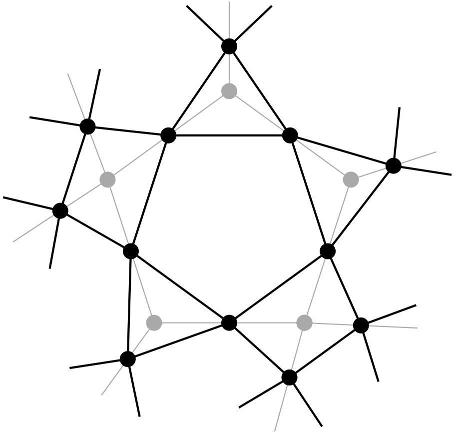

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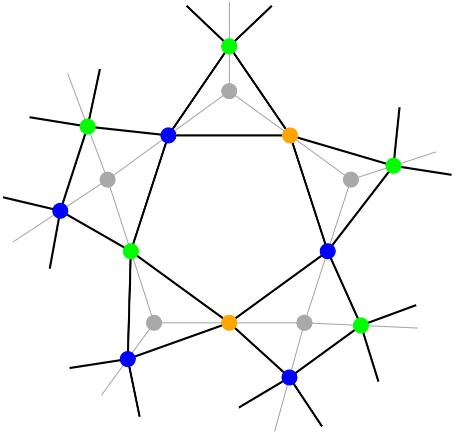

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- Medial graphs of plane graphs are:
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- Problem 1 reduces to investigating 3-colorability of planar graphs with maximum degree 4;
- Deciding whether a planar graph G with $\Delta(G) = 4$ admits a 3-coloring is NP-complete [8];
- $\rightarrow$ Lots of attention given to 3-colorability.

3-Colorability of Planar Graphs

Theorem 2 (Heawood [11])

A plane triangulation is 3-colorable if and only if all its vertices have even degree.

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- Improved by Grünbaum (and Aksenov) to planar graphs with at most three triangles.

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- What if we allow many triangles in planar graphs?

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Conjecture 4 (Havel [10])

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Conjecture 5 (Steinberg [12])

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- Disproved by Cohen-Addad, Hebdige, Kráľ, Li, and Salgado [2].

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- Open: Are planar graphs without cycles of lengths from 4 to 7 (or even 6) 3-choosable?

Our Result

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Every loopless planar graph with maximum degree 4 obtained as a subgraph of the medial graph of a bipartite plane graph is 3-choosable.

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- Answer to Problem 1 also in the list setting.

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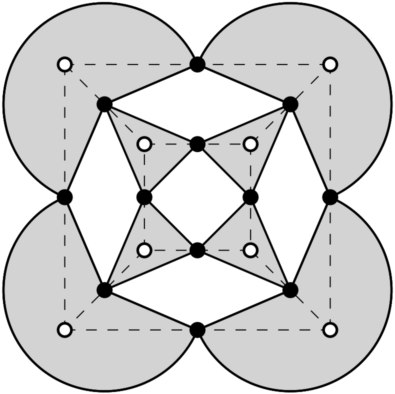
  - Triangles are close & there are short cycles $\rightarrow$ still 3-choosable!

Sketch of Proof – 2



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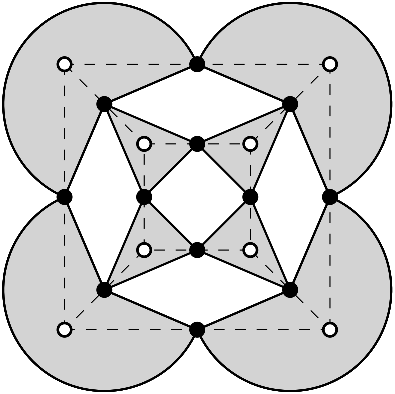

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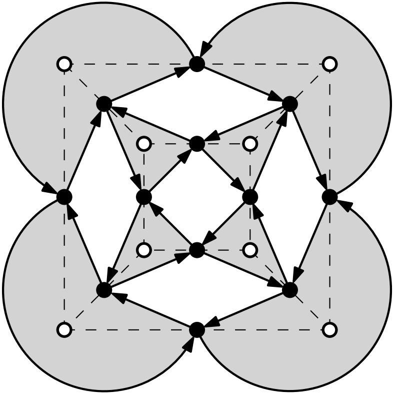

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Let D be a directed graph, and let L be a list-assignment such that $|L(v)| \geq d_D^+(v) + 1$ for each $v \in V(D)$. If $E^e(D) \neq E^o(D)$, then D is L-colorable.

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- We need to prove that the number of even spanning Eulerian subgraphs is different from the number of odd spanning Eulerian subgraphs in G.

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  - First, color the faces of H properly with two colors (possible since the dual of H is bipartite);

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- For a subgraph X of G, we define:

$$\partial_X(H) = \partial(H) \cap X,\ \mathrm{int}_X(H) = \mathrm{int}(H) \cap X,\ \mathrm{ext}_X(H) = \mathrm{ext}(H) \cap X.$$

Sketch of Proof – 5

Observation 1

Let D_1 and D_2 be two directed cycles in G intersecting (i.e., having some common vertices) in such a way that $\partial(D_2) \cap \mathrm{int}(D_1) \neq \emptyset$ and $\partial(D_2) \cap \mathrm{ext}(D_1) \neq \emptyset$. Then $E(D_1) \cap E(D_2) \neq \emptyset$.

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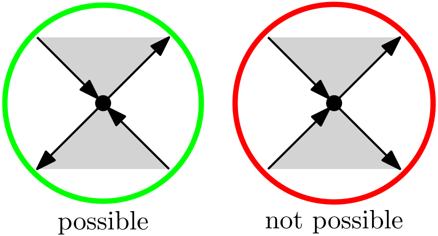

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- $\Rightarrow$ We distinguish two types of directed cycles in G:
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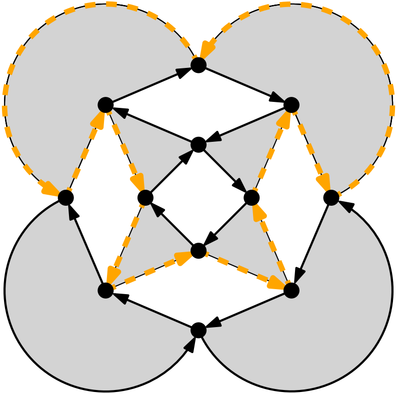

Sketch of Proof – 7

- For a cycle D, the D-complement of a spanning Eulerian subgraph X of G is the spanning Eulerian subgraph $\overline{X}^D$ with the edge set

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- $\overline{X}^D$ is also Eulerian by Observation 1.

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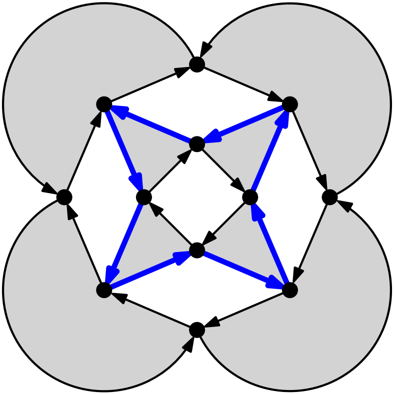

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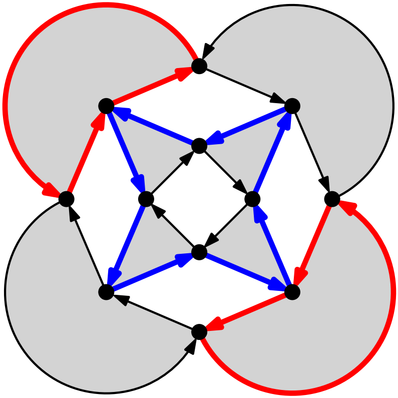

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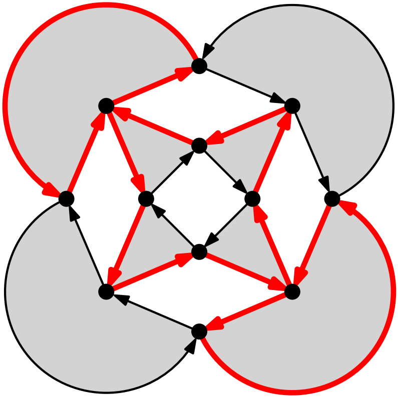

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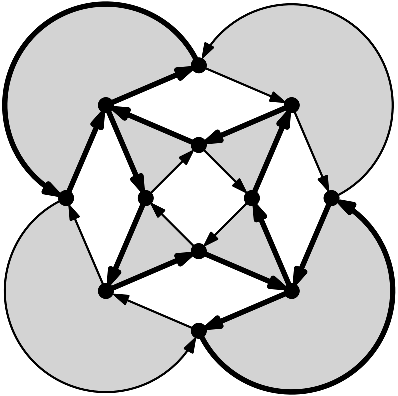

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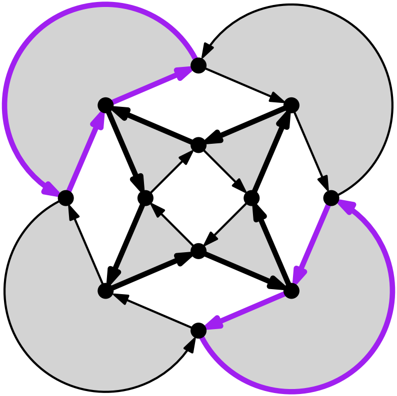

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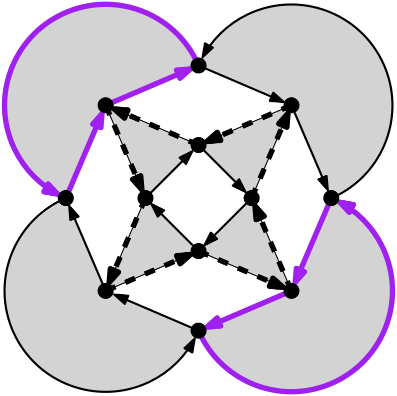

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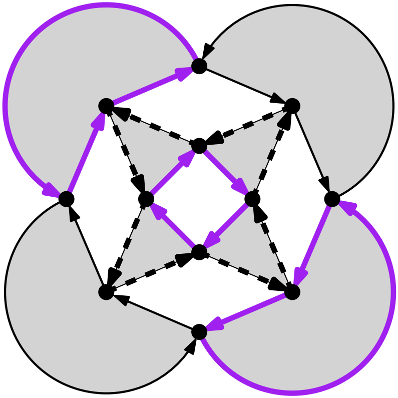

Sketch of Proof – 8

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For an odd black cycle D, the D-complement of an odd (even) Eulerian spanning subgraph X is an even (odd) Eulerian spanning subgraph $\overline{X}^D$.

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Claim 2

Let X be an Eulerian spanning subgraph of G, and let D be a white odd Eulerian subgraph of X. Then, there is an odd black cycle in $\mathrm{int}_X(D)$ or $\mathrm{int}_{\overline{X}^D}(D)$.

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- If in the step i we remove from $\mathcal{E}$ some X, then we also remove its C_i-complement;
- Such pairs are always removed at the same step:

Claim 3

The number of odd Eulerian spanning subgraphs removed from $\mathcal{E}$ at step i is equal to the number of even such subgraphs.

Sketch of Proof – 10

- After all cycles from $\mathcal{O}$ are removed, there is no odd Eulerian spanning subgraph left in $\mathcal{E}$;

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White faces of G can be colored with two colors, red and blue, such that every odd black cycle shares an edge with the boundary of at least one red and at least one blue face.

- $\Rightarrow$ There is at least one even Eulerian spanning subgraph, containing at least one edge of every odd cycle in G, but not all edges of any!

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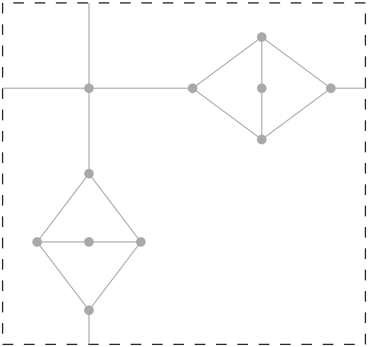

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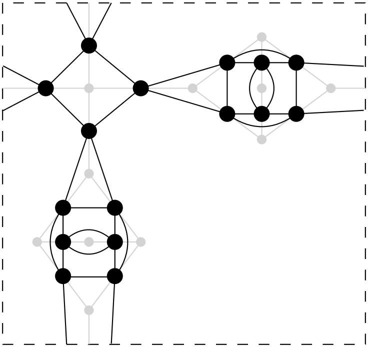

Further Work

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Every simple plane graph whose faces can be properly colored with two colors such that one color class contains only even faces is 3-colorable.

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- Why no parallel edges?
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Question 10

Is every simple plane graph whose faces can be properly colored with two colors such that one color class contains only even faces also 3-choosable?

Further Discussion

Theorem 11 (Ellingham & Goddyn [7])

If G is a d-regular d-edge-colorable planar multigraph, then G is d-edge-choosable.

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- Coloring the edges of the graph properly, but...
- medial graphs in cubic planar graphs are exactly their line graphs.
- Such graphs have two types of faces – one type are just triangles.

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Question 13 (Kaiser)

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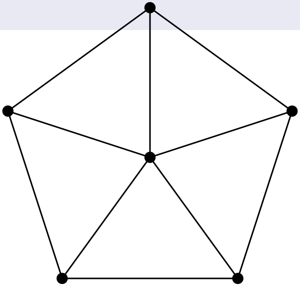

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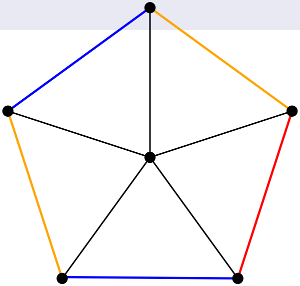

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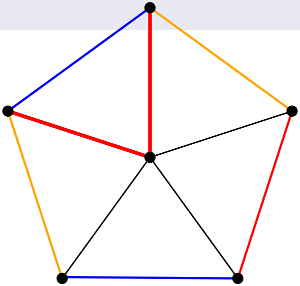

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Thank you!