Colorful Graph Theory

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19. Konferencia košických matematikov

Herľany, April 13, 2018

Sunway TaihuLight

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- Ranked #1 in the TOP500 list in March 2018 as the fastest supercomputer
- 93 petaflops $= 93 \cdot 10^{15}$ flops (floating point operations per second)
- 10,649,600 CPU cores

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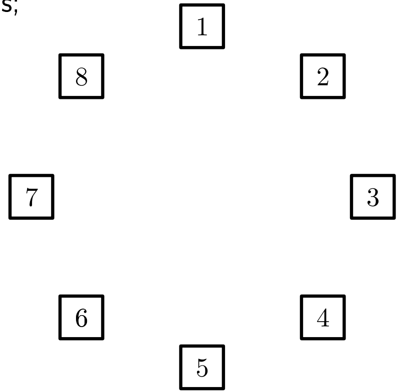

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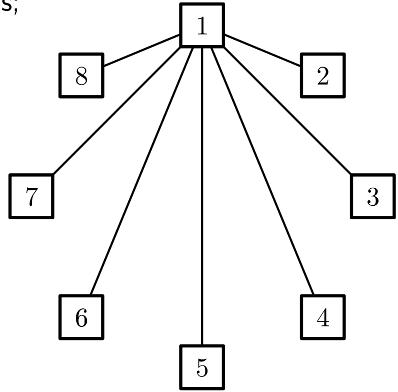

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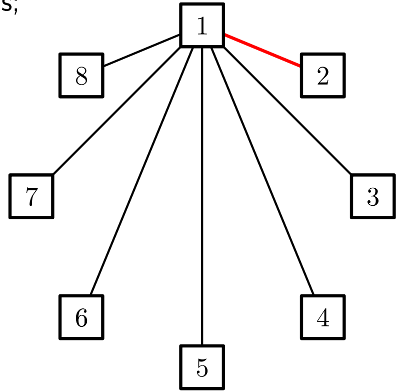

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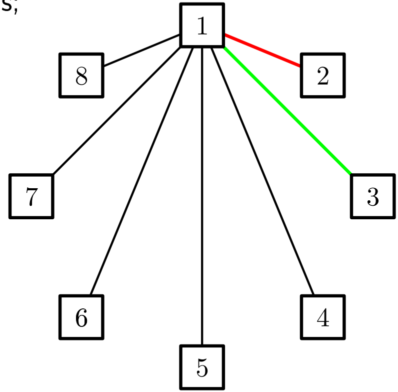

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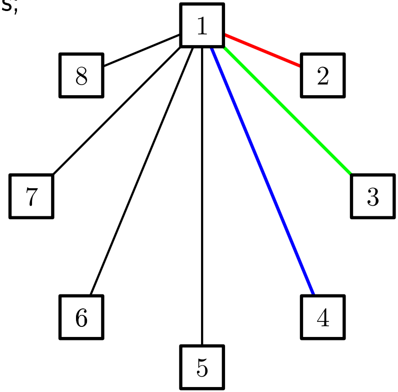

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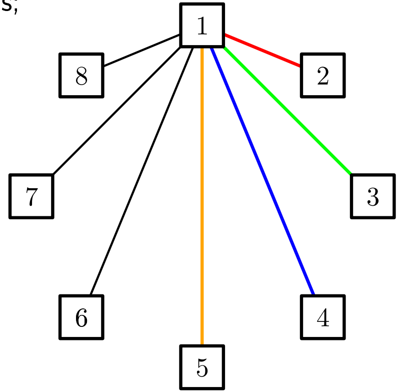

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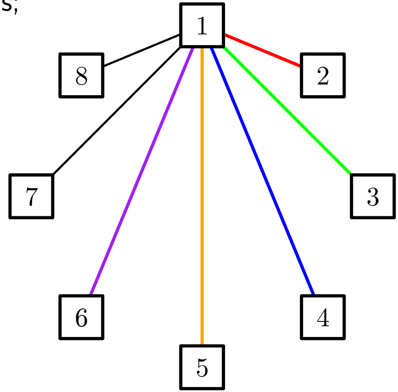

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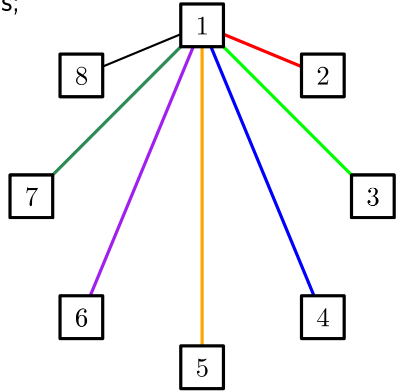

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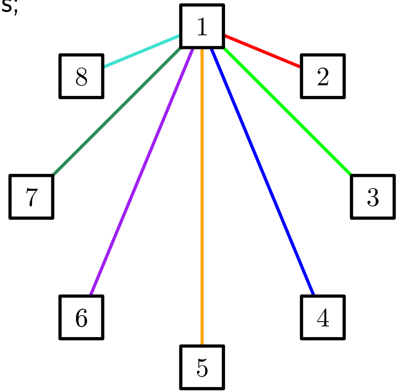

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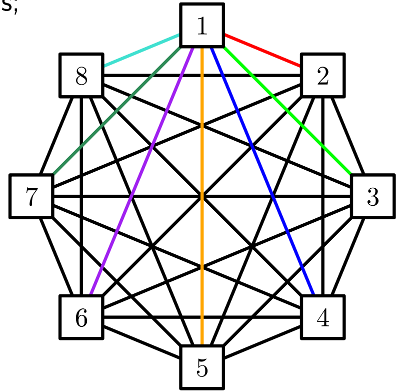

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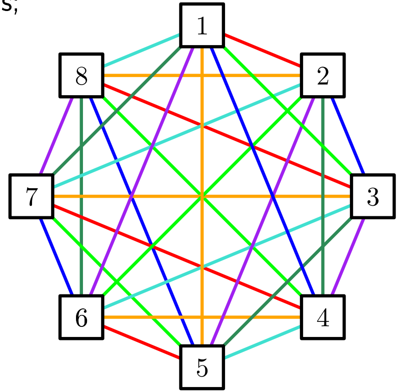

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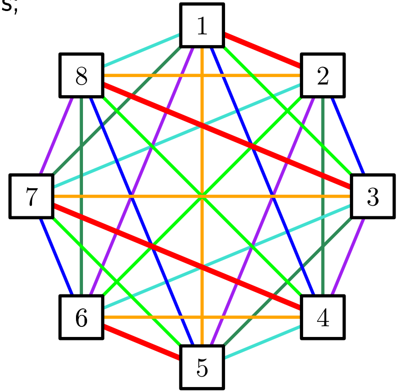

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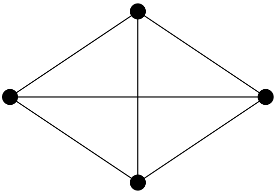

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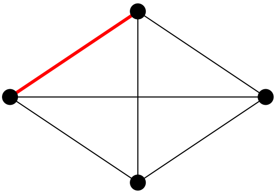

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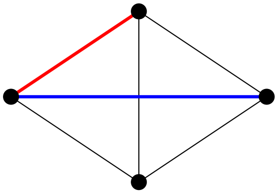

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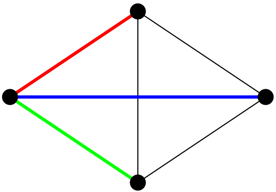

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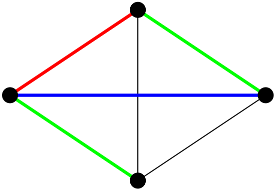

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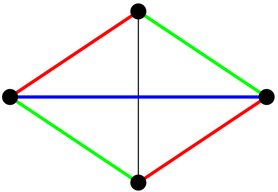

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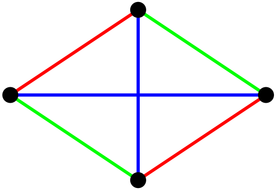

$\longrightarrow \quad \chi'(K_4) = 3$

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Theorem 1 (Vizing, 1964)

For every (simple) graph G

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

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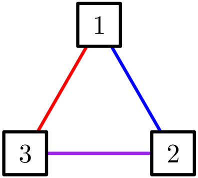

Bipartite and complete graphs

Theorem 2 (König, 1916)

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- For complete graphs K_{2k}, we have $2k-1$ disjoint perfect matchings; we assign the same color to all edges of a matching, so:

  $$\chi'(K_{2k}) = \Delta(K_{2k}) = 2k-1.$$

  Complete graphs of odd order need additional color:

  $$\chi'(K_{2k+1}) = \Delta(K_{2k+1}) + 1 = 2k+1.$$

Adding assumptions

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- We will focus on three types:
  - Acyclic edge-coloring;
  - Strong edge-coloring;
  - Locally irregular edge-coloring.

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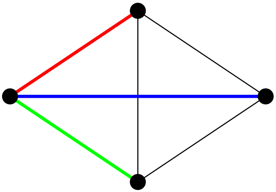

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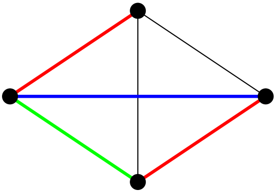

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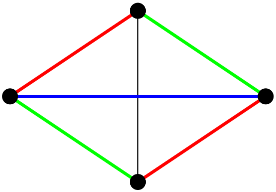

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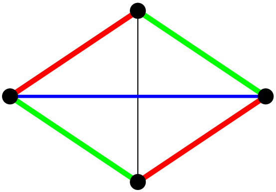

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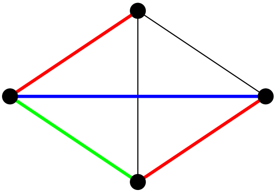

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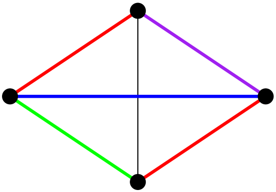

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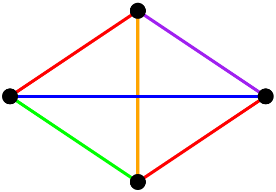

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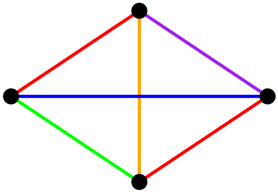

- So, $\chi'_a(K_4) = 5 = \Delta(K_4) + 2$

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Conjecture 3 (Fiamčík, 1978; Alon, Sudakov, Zaks, 2001)

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- Conjecture 3 is not confirmed even for complete graphs!

Perfect 1-factorization

Conjecture 4 (Kotzig, 1964)

For every $n \geq 2$, K_{2n} can be decomposed into $2n-1$ perfect matchings such that the union of any two matchings forms a hamiltonian cycle in K_{2n}.

- Closely related to acyclic edge-colorings.
- If the Conjecture 4 is true, the removal of one vertex from K_{2n} results in an acyclic edge coloring of K_{2n-1} with $2n-1 = \Delta(K_{2n-1}) + 1$ colors, which is optimal.

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- If K_{n+1} has perfect 1-factorization, then $K_{n,n}$ has it also.

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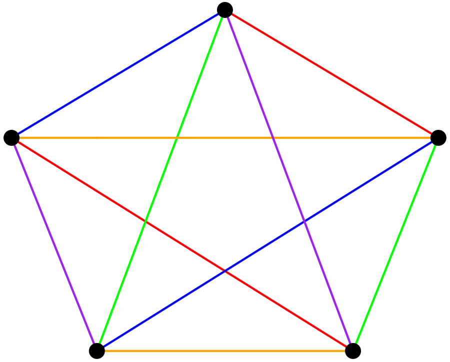

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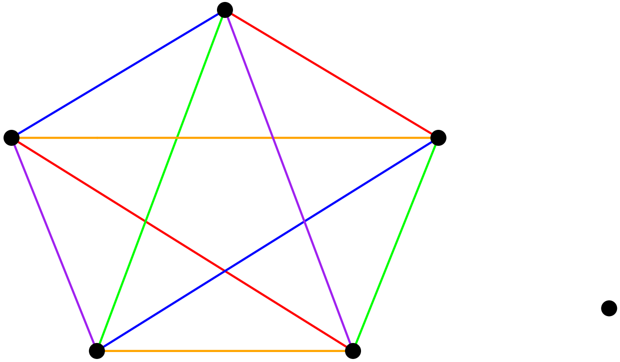

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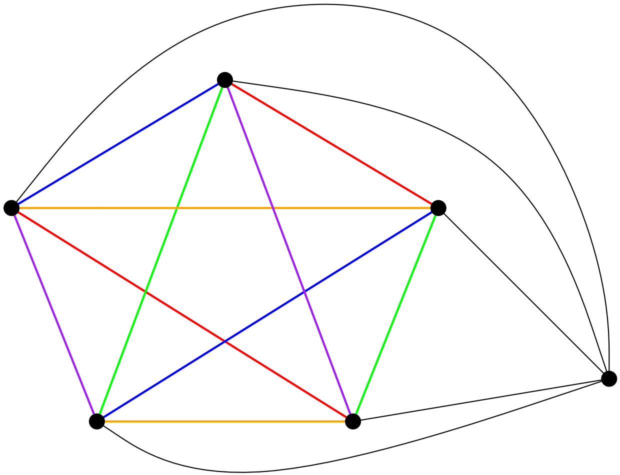

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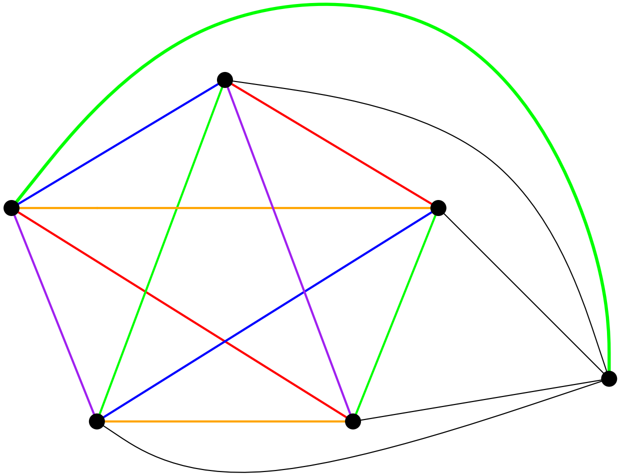

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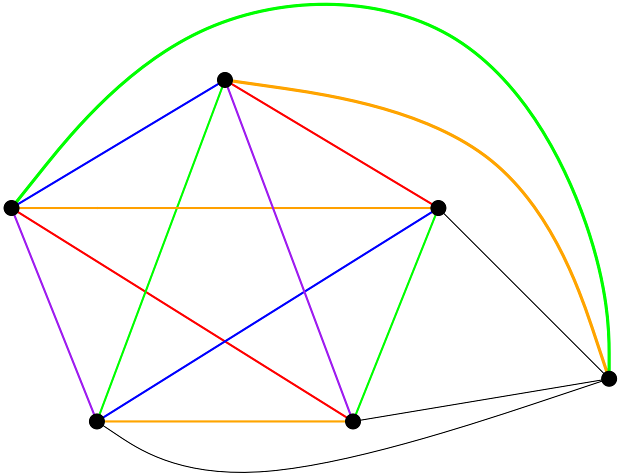

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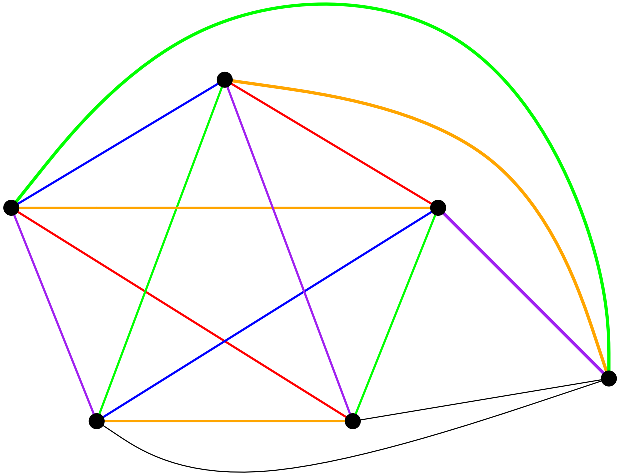

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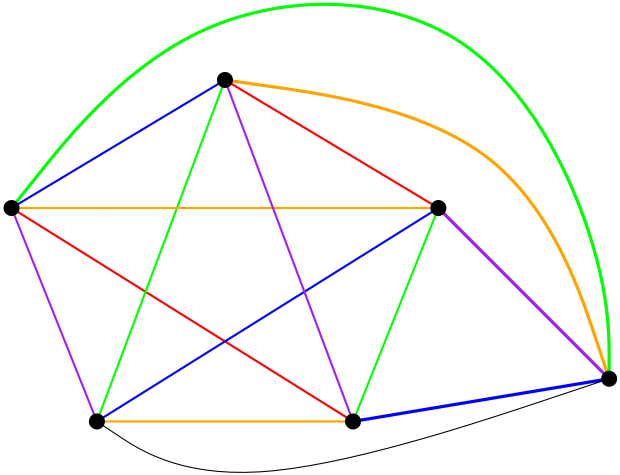

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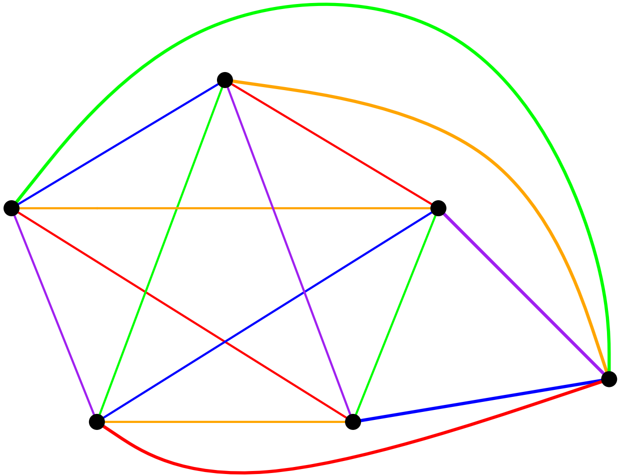

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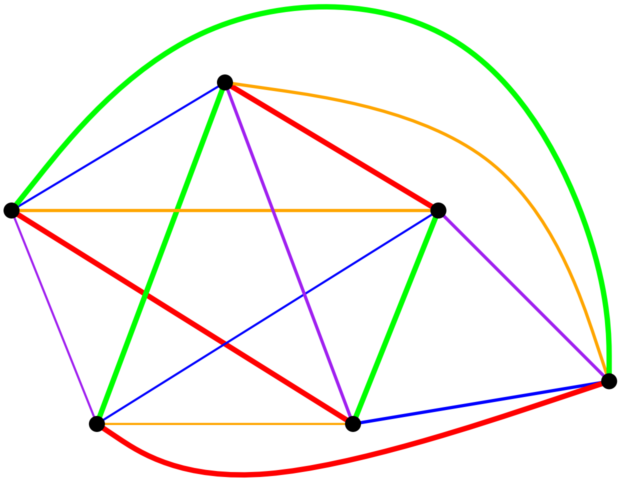

General graphs

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Theorem 6 (Alon, Sudakov, Zaks, 2001)

For every graph G with girth at least $C\Delta(G) \log \Delta(G)$, for a constant C, it holds

$$\chi'_a(G) \leq \Delta(G) + 2\,.$$

Subcubic graphs

- The notion of acyclic colorings was first introduced in 1973 by Grűnbaum for the vertex version. In 1979, Burnstein proved that 5 colors suffice for acyclic vertex coloring of every graph G with $\Delta(G) \leq 4$.

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- The maximum degree of the line graph $L(G)$ of a subcubic graph G is at most 4...

Corollary 7 (Burnstein, 1979)

Let G be a subcubic graph. Then

$$\chi'_a(G) \leq 5\,.$$

Planar graphs

- Another candidate class of graphs to confirm Conjecture 4

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Theorem 8 (Wang, Zhang, 2017+)

Let G be a planar graph. Then

$$\chi'_a(G) \leq \Delta(G) + 6\,.$$

Planar graphs - Δ colors

- Cohen, Havet and Müller conjectured that every planar graph G with large enough maximum degree has $\chi'_a(G) = \Delta$ (note the analogy to Vizing's conjecture)

Planar graphs - Δ colors

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Theorem 9 (Cranston, 2017+)

Let G be a planar graph with $\Delta(G) \geq 4.2 \cdot 10^{14}$. Then

$$\chi'_a(G) = \Delta(G) .$$

Planar graphs - Δ colors

- We considered the problem with the girth condition added.

Planar graphs - Δ colors

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Theorem 10 (Húdak et al., 2012)

Let G be a planar graph with girth g and maximum degree Δ. Then $\chi'_a(G) = \Delta$ if one of the following conditions holds:

- $\Delta \geq 3$ *and* $g \geq 12$, *or*
- $\Delta \geq 4$ *and* $g \geq 8$, *or*
- $\Delta \geq 5$ *and* $g \geq 7$, *or*
- $\Delta \geq 6$ *and* $g \geq 6$, *or*
- $\Delta \geq 10$ *and* $g \geq 5$.

Strong edge-coloring

Definition

- Distance between edges: distance between corresponding vertices in the line graph (adjacent edges are at distance 1)

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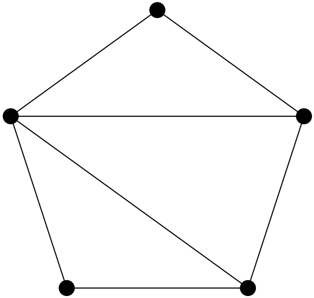
- The smallest k for which G admits a strong k-edge-coloring is the strong chromatic index of G, $\chi'_s(G)$.
- Strong edge coloring of G is a *vertex 2-distance coloring* of its line graph $L(G)$

$$\chi'_s(G) = \chi(L(G)^2) .$$

Example



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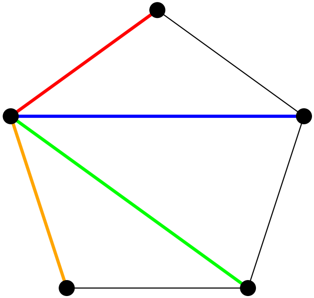

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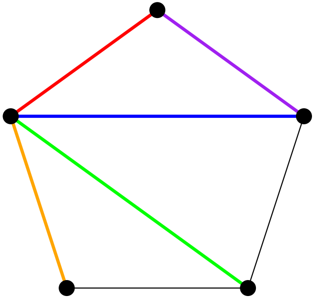

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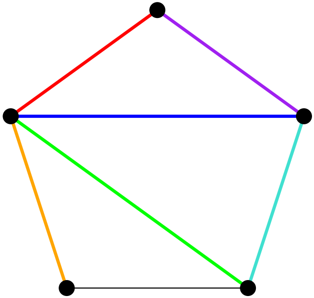

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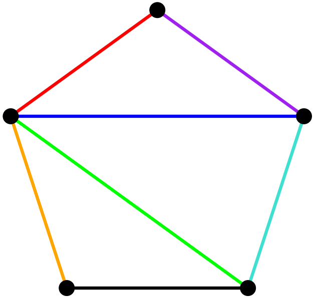

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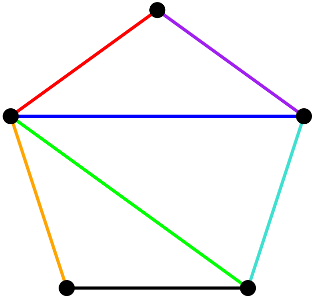

$$\chi'_s(G) = 7$$

The Conjecture

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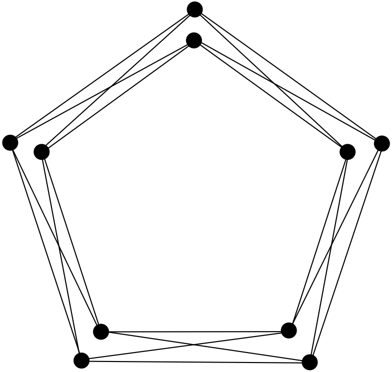
Conjecture 11 (Erdős, Nešetřil, 1985)

For every graph G it holds

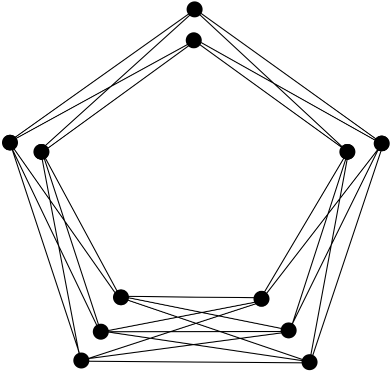
$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta(G)^2\,, & \Delta(G) \text{ is even;} \\ \frac{1}{4}(5\Delta(G)^2 - 2\Delta(G) + 1)\,, & \Delta(G) \text{ is odd.} \end{cases}$$

The Conjecture

- The bounds in Conjecture 11 are tight for every Δ:



$\Delta(G) = 4$



$\Delta(G) = 5$

The Conjecture

The construction of graphs achieving the conjectured bound:

- For even Δ replace each vertex of a 5-cycle with $\frac{\Delta}{2}$ vertices;
- For odd Δ replace two consecutive vertices of a 5-cycle with $\frac{\Delta+1}{2}$ vertices and the others with $\frac{\Delta-1}{2}$ vertices.

General graphs

- By greedy method, we have $\chi'_s(G) \leq 2\Delta(G)(\Delta(G) - 1) + 1$

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Theorem 12 (Bonamy, Perret, Postle, 2017+)

For every graph G with sufficiently large maximum degree it holds

$$\chi'_s(G) \leq 1.835\Delta(G)^2$$

Subcubic & Subquartic graphs

Theorem 13 (Andersen, 1992)

Let G be a graph with $\Delta(G) = 3$. Then,

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Theorem 14 (Cranston, 2006)

Let G be a graph with $\Delta(G) = 4$. Then,

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Bipartite graphs

Conjecture 15 (Faudree et al., 1990)

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Conjecture 16 (Brualdi, Quinn Massey, 1993)

If G is bipartite graph with maximum degree of partite sets Δ_1 and Δ_2, then

$$\chi'_s(G) \leq \Delta_1 \cdot \Delta_2 \, .$$

Bipartite graphs

Theorem 17 (Steger, Yu, 1993)

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- For $(3, \Delta)$-graphs there is a weaker result: $\chi'_s(G) \leq 4\Delta$

Planar graphs

Theorem 19 (Faudree et al., 1990)

Let G be a planar graph. Then,

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- Altogether we need $4\ \chi'(G)$ colors

Planar graphs

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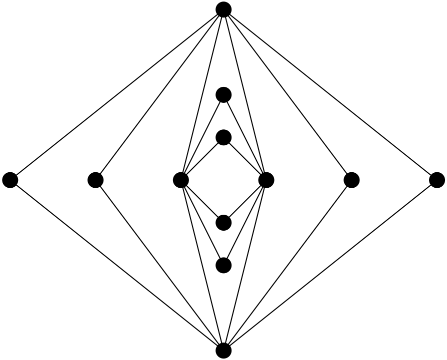

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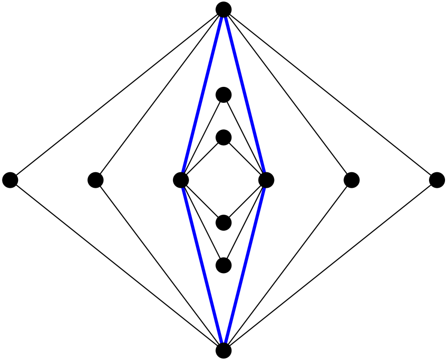

Planar graphs

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Conjecture 20 (Hudák et al., 2014)

There exists a constant C such that for every planar graph G with girth $g \geq 5$ it holds

$$\chi'_s(G) \leq \left\lceil \frac{2g(\Delta(G) - 1)}{g - 1} \right\rceil + C$$

Locally irregular edge-coloring

Basics

- A graph G is locally irregular if every two adjacent vertices have distinct degrees.
- An edge-coloring is locally irregular if every color class induces a locally irregular graph.

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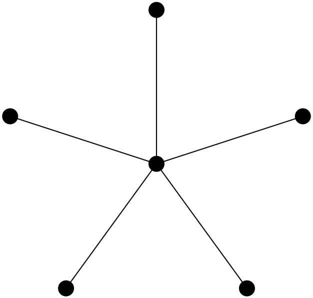
- Always improper — paths of odd length do not admit such a coloring
- Introduced by Baudon, Bensmail, Przybyło, and Woźniak in 2013 (the paper published in 2015).
- Motivated by the (1-2-3)-Conjecture:

  For every graph with no K_2 component there exists an edge weighting with 1, 2, and 3 such that for every two adjacent vertices the sums on their incident edges are distinct.

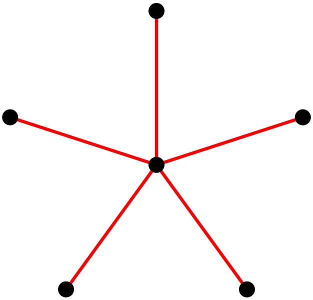
Example: K_5

- A test for the audience... How many colors?



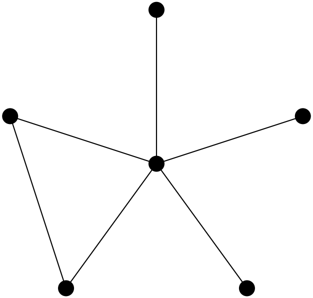
Example: K_5

- Correct!



Example: K_5

- And now — add one edge:



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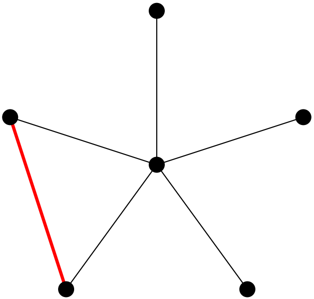

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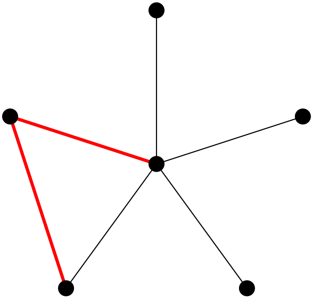

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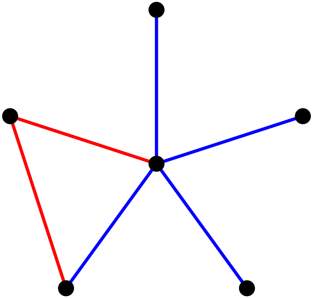

Decomposable graphs

- A graph is decomposable if it admits a locally irregular edge-coloring (LIE-C).
- The minimum k for which there is a LIE-C of a graph G with k colors is the locally irregular chromatic index of G, $\chi'_{\mathrm{irr}}(G)$.

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- The minimum k for which there is a LIE-C of a graph G with k colors is the locally irregular chromatic index of G, $\chi'_{\mathrm{irr}}(G)$.
- Not all graphs are decomposable, e.g. odd-length paths, odd-length cycles.
- A complete characterization was given by Baudon, Bensmail, Przybyło, and Woźniak.

Decomposable graphs

Define a family of graphs $\mathcal{T}$ recursively:

- The triangle C_3 belongs to $\mathcal{T}$.
- Every other graph of this family can be constructed by taking an auxiliary graph F which might either be a path of even length or a path of odd length with a triangle glued to one end, then choosing a graph $G \in \mathcal{T}$ containing a triangle with at least one vertex v of degree 2 and finally identifying v with a vertex of degree 1 in F.

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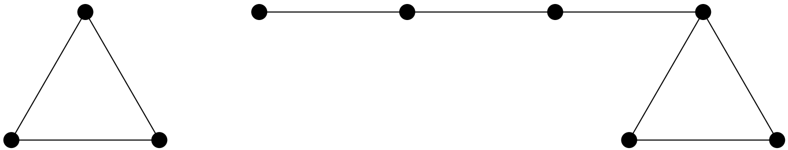

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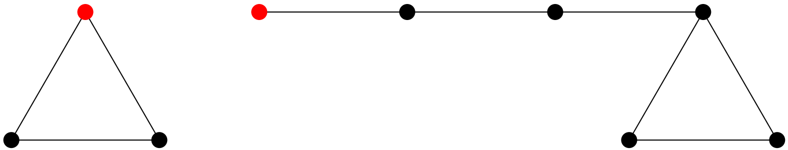

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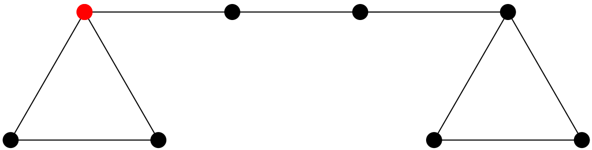

The Conjecture

Conjecture 21 (Baudon et al., 2015)

For every decomposable graph G, it holds $\chi'_{\mathrm{irr}}(G) \leq 3$.

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Theorem 22 (Baudon et al., 2015)

For every d-regular graph G, with $d \geq 10^7$, it holds $\chi'_{\mathrm{irr}}(G) \leq 3$.

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Theorem 22 (Baudon et al., 2015)

For every d-regular graph G, with $d \geq 10^7$, it holds $\chi'_{\mathrm{irr}}(G) \leq 3$.

Theorem 23 (Przybyło, 2016)

For every graph G, with $\delta(G) \geq 10^{10}$, it holds $\chi'_{\mathrm{irr}}(G) \leq 3$.

The upper bound

Bensmail, Merker, and Thomassen established the first constant upper bound using decompositions into bipartite graphs.

Theorem 24 (Bensmail et al., 2017)

For every decomposable graph G, it holds $\chi'_{\mathrm{irr}}(G) \leq 328$.

Currently the best:

Theorem 25 (BL, Przybyło, Soták, 2018+)

For every decomposable graph G, it holds $\chi'_{\mathrm{irr}}(G) \leq 220$.

Subcubic graphs

Theorem 26 (BL, Przybyło, Soták, 2018+)

For every decomposable graph G with $\Delta(G) = 3$, it holds $\chi'_{\mathrm{irr}}(G) \leq 4$.

Bipartite graphs

Theorem 27 (Baudon et al., 2015)

Let G be a regular bipartite graph with minimum degree at least 3. Then

$$\chi'_{\mathrm{irr}}(G) \leq 2\,.$$

A decomposable bipartite graph is balanced if all the vertices in one of the two partition parts have even degrees.

Lemma 28 (Bensmail et al., 2017)

Let F be a balanced forest. Then F admits a LIE-C with at most 2 colors. Moreover, for each vertex v in the partition with no vertex of odd degree, all edges incident to v have the same color.

Bipartite graphs

Theorem 29 (BL, Przybyło, Soták, 2018+)

Let G be a (multi)graph not isomorphic to an odd cycle. Then

$$\chi'_{\mathrm{irr}}(\mathcal{S}(G)) \leq 2\,.$$

Here, $\mathcal{S}(G)$ denotes the full subdivision of G, i.e. each edge of G is subdivided once.

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Question 30

Is every connected balanced graph, which is not a cycle of length $4k+2$, locally irregularly 2-edge-colorable?

Bipartite graphs

Theorem 31 (BL, Przybyło, Soták, 2018+)

Let G be a balanced graph. Then

$$\chi'_{\mathrm{irr}}(G) \leq 4\,.$$

Theorem 32 (BL, Przybyło, Soták, 2018+)

Let G be a decomposable bipartite graph. Then

$$\chi'_{\mathrm{irr}}(G) \leq 7\,.$$

Moreover, if G has an even number of edges, then the upper bound is 6.

Ďakujem!