

On 3-Choosability of Planar Graphs with Maximum Degree 4

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joint work with
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Graph Theory Seminar

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The Problem

Problem 1 (Czap, Jendroľ & Voigt [3])

Is there a bipartite plane graph such that its medial graph has chromatic number 4?

In other words:

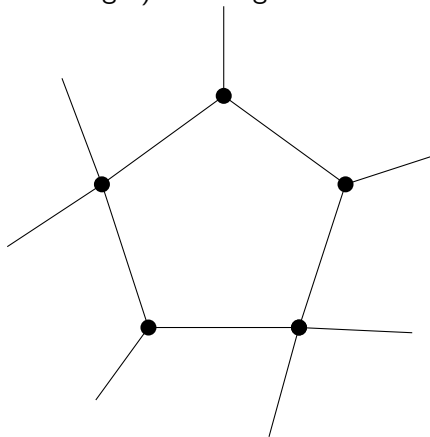
Is there a bipartite plane graph that needs 4 colors for facially-proper edge-coloring?

Facially-Proper Edge-Coloring

- **Facially-proper edge-coloring** of a plane graph is a coloring with edges consecutive on some facial trail (i.e., **facially-adjacent** edges) receiving distinct colors.

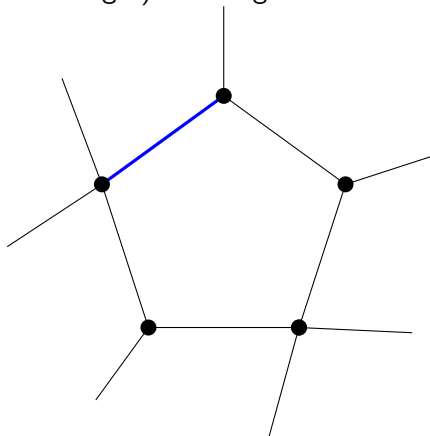
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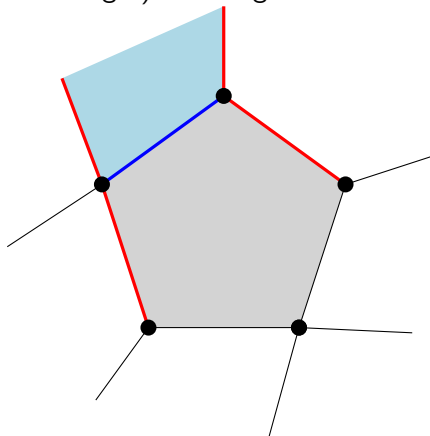
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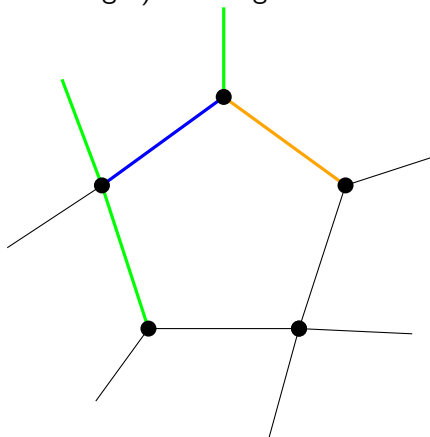
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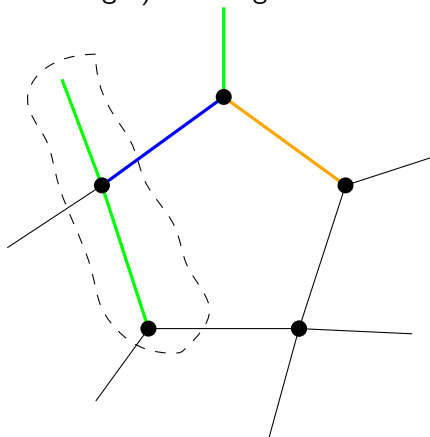
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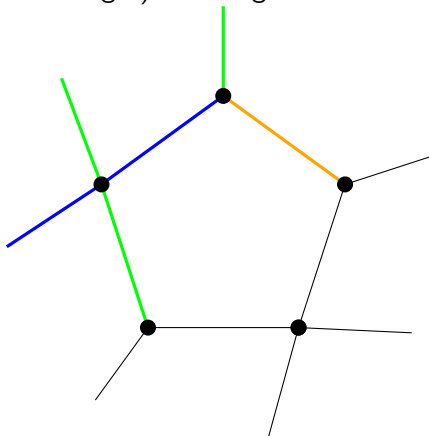
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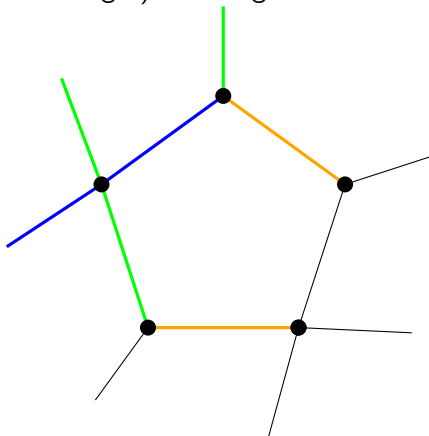
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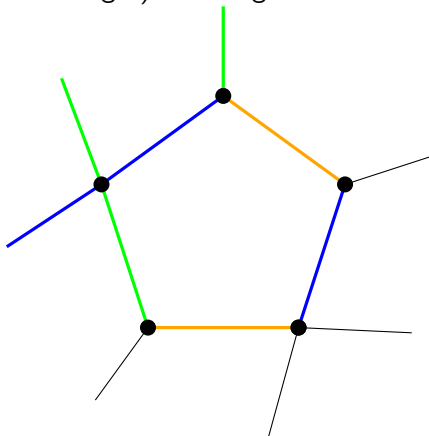
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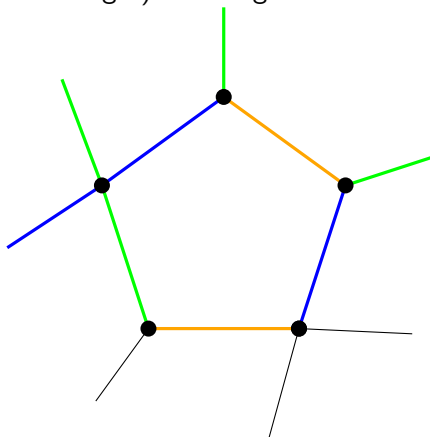
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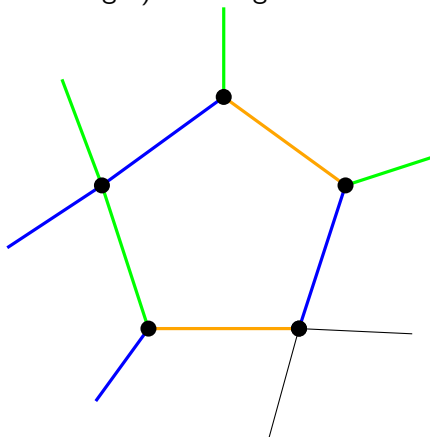
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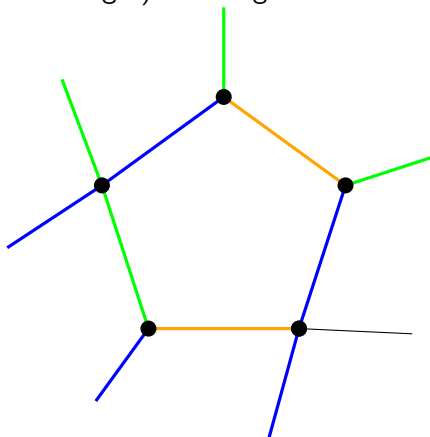
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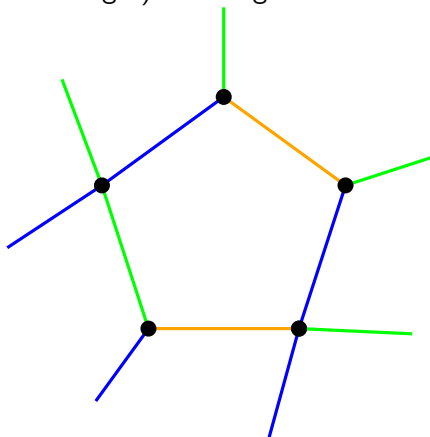
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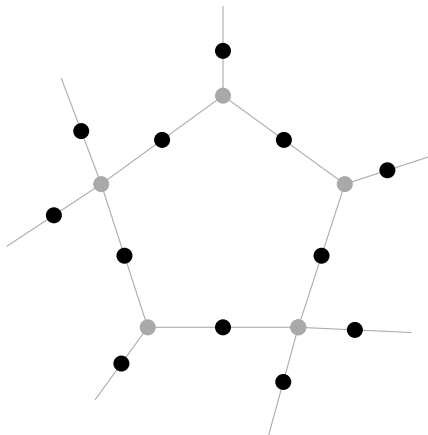


Medial Graph

- The medial graph $M(G)$ of a plane graph G :
 - $V(M(G)) = E(G)$;
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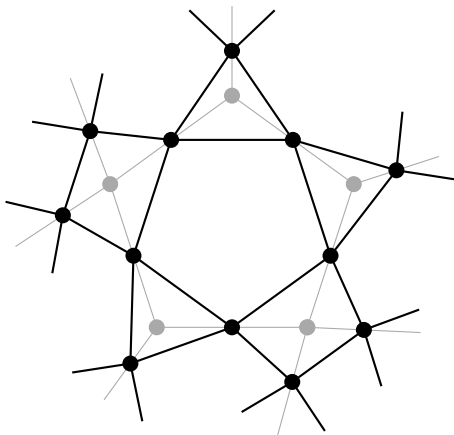
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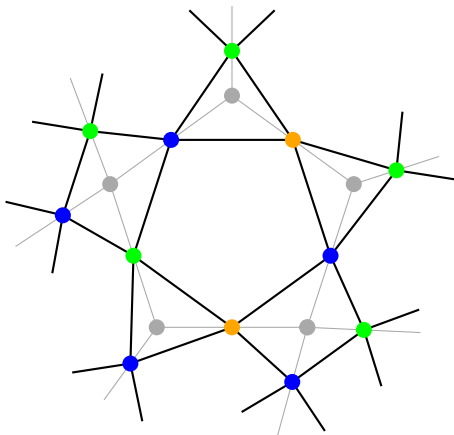
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- Problem 1 reduces to investigating 3-colorability of planar graphs with maximum degree 4;
- Deciding whether a planar graph G with $\Delta(G) = 4$ admits a 3-coloring is NP-complete [7];
- \rightarrow Lots of attention given to 3-colorability.

3-Colorability of Planar Graphs

Theorem 2 (Heawood [10])

A plane triangulation is 3-colorable if and only if all its vertices have even degree.

- With many generalizations...

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- Improved by Grünbaum (and Aksenov) to planar graphs with at most three triangles.

3-Colorability of Planar Graphs with Δ 's

- What if we **allow many triangles** in planar graphs?

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Conjecture 4 (Havel [9])

There exists an absolute constant d such that if G is a planar graph and every two distinct triangles in G are at distance at least d , then G is 3-colorable.

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- **Disproved** by Cohen-Addad, Hebdige, Král', Li, and Salgado [2].

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- Open: Are planar graphs without cycles of lengths from 4 to 7 (or even 6) 3-choosable?

Our Result

Theorem 7 (Dross, BL, Maceková & Soták – 2018⁺)

Every loopless planar graph with maximum degree 4 obtained as a subgraph of the medial graph of a bipartite plane graph is 3-choosable.

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- Answer to Problem 1 also in the list setting.

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- Structure of our graph:
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 - \Rightarrow The faces are properly 2-colored;

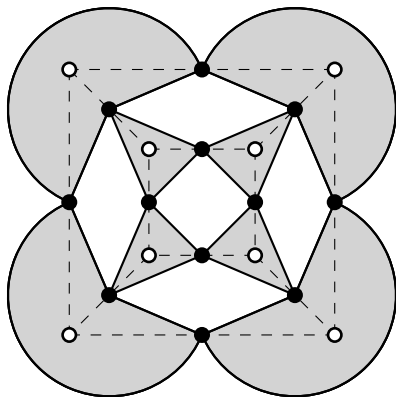
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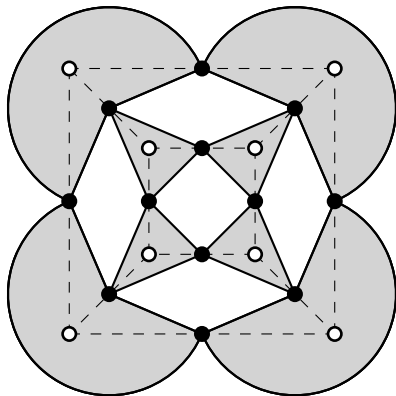
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 - Triangles are close & there are short cycles \rightarrow still 3-choosable!

Sketch of Proof – 2

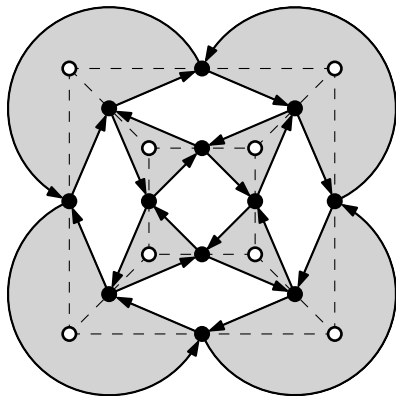


Sketch of Proof – 2



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Theorem 8 (Alon & Tarsi [1])

Let D be a directed graph, and let L be a list-assignment such that $|L(v)| \geq d_D^+(v) + 1$ for each $v \in V(D)$. If $E^e(D) \neq E^o(D)$, then D is L -colorable.

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- We need to **prove that the number of even spanning Eulerian subgraphs is different from the number of odd spanning Eulerian subgraphs in G .**

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- For a subgraph X of G , we define:

$$\partial_X(H) = \partial(H) \cap X, \text{int}_X(H) = \text{int}(H) \cap X, \text{ext}_X(H) = \text{ext}(H) \cap X.$$

Sketch of Proof – 5

Observation 1

Let D_1 and D_2 be two directed cycles in G intersecting (i.e., having some common vertices) in such a way that $\partial(D_2) \cap \text{int}(D_1) \neq \emptyset$ and $\partial(D_2) \cap \text{ext}(D_1) \neq \emptyset$. Then $E(D_1) \cap E(D_2) \neq \emptyset$.

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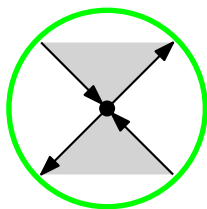
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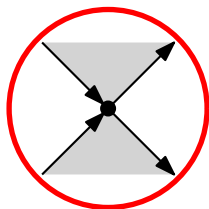
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Sketch of Proof – 6

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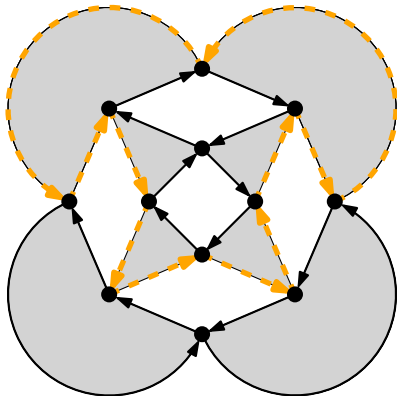
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Sketch of Proof – 7

- For a cycle D , the D -complement of a spanning Eulerian subgraph X of G is the spanning Eulerian subgraph \bar{X}^D with the edge set

$$E(\bar{X}^D) = E(\text{ext}_X(D)) \cup E(\text{int}_{\bar{X}}(D)) \cup E(\partial_{\bar{X}}(D));$$

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- \bar{X}^D is also Eulerian by Observation 1.

Sketch of Proof – 8

Claim 1

For an odd black cycle D , the D -complement of an odd (even) Eulerian spanning subgraph X is an even (odd) Eulerian spanning subgraph \bar{X}^D .

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Claim 2

Let X be an Eulerian spanning subgraph of G , and let D be a white odd Eulerian subgraph of X . Then, there is an odd black cycle in $\text{int}_X(D)$ or $\text{int}_{\bar{X}^D}(D)$.

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- If in the step i we remove from \mathcal{E} some X , then we also remove its C_i -complement;

Sketch of Proof – 9

- \mathcal{E} – the set of all Eulerian spanning subgraphs of G ;
- \mathcal{O} – a sorted set of all odd black cycles in G (sorted in ascending order by number of faces in their interiors);
- Suppose there are k cycles, C_1, C_2, \dots, C_k , in \mathcal{O} ;
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- Such pairs are always removed at the same step:

Claim 3

The number of odd Eulerian spanning subgraphs removed from \mathcal{E} at step i is equal to the number of even such subgraphs.

Sketch of Proof – 10

- After all cycles from \mathcal{O} are removed, there is **no odd Eulerian spanning subgraph** left in \mathcal{E} ;

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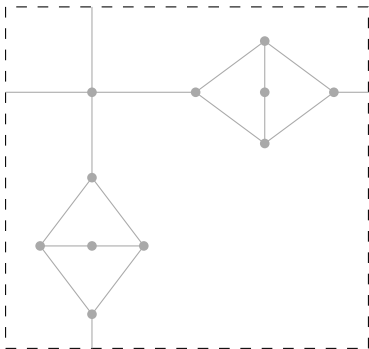
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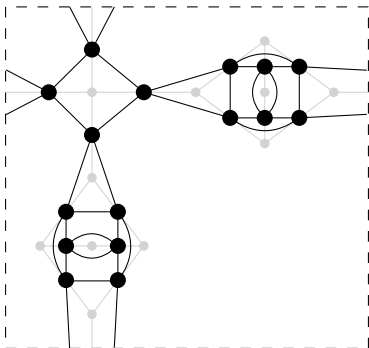
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Further Work

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Question 10

Is every simple plane graph whose faces can be properly colored with two colors such that one color class contains only even faces also 3-choosable?

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Thank you!