Edge-colorings as edge-decompositions

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joint work with
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Cycles and Colourings
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Odd edge-coloring

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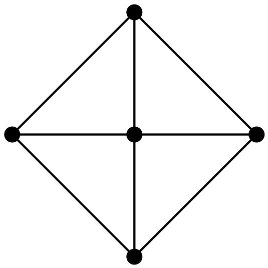

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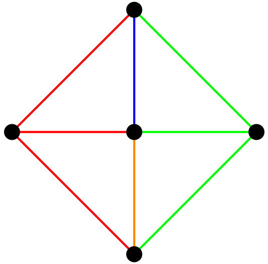

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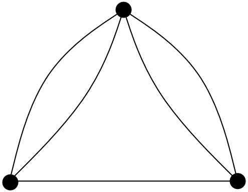
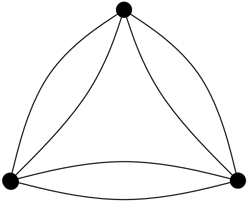
- Motivation: facial-parity edge-coloring [6] (click here for more details)

Multigraphs

- There are graphs with $\chi'_o > 4$:





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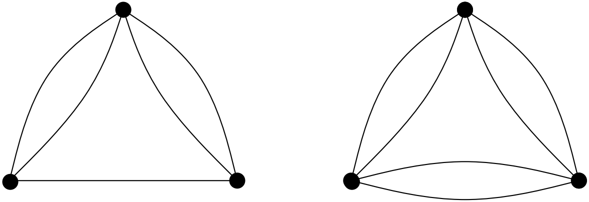

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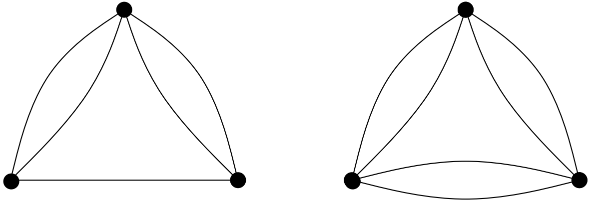

- Shannon's triangle is a loopless graph on three pairwise adjacent vertices;
- Let p, q, r be the parities of the multiplicities (2 for even, 1 for odd) of the edges: we then say that Shannon's triangle is of type (p, q, r);

Multigraphs

Theorem 2 (BL, Petruševski, Škrekovski [3])

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- Hence, in general we know a lot...
  What about prescribing parities to the vertices instead of taking odd subgraphs?

Vertex-parity edge-coloring

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- A vertex-parity edge-coloring of a parity pair (G, π) is a (not necessarily proper) edge-coloring such that at every vertex v each appearing color c is in parity accordance with π, i.e. the number of edges of color c incident to v is even if $\pi(v) = 0$, and odd if $\pi(v) = 1$.

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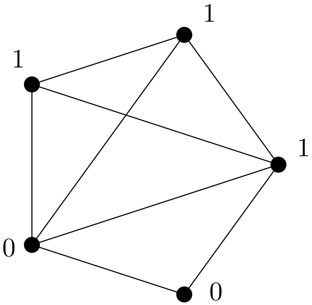
- $\chi'_p(G,\pi)$ - the vertex-parity chromatic index
- Necessary conditions for the existence of $\chi'_p(G,\pi)$:
  - (P_1) Every vertex v of (G,π) with $\pi(v) = 0$ has even degree in G.
  - (P_2) In every component of G, there are zero or at least two vertices with the vertex signature value 1.
- A parity pair satisfying (P_1) and (P_2) is a proper parity pair.

Example



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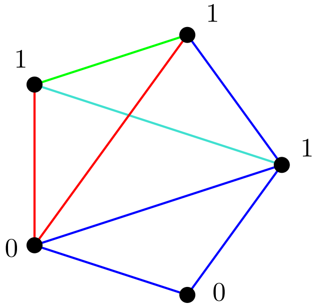

Results

- Given a graph G_0 and a pair (G, π), we say (G, π) is a derivative of G_0, denoted by $(G, \pi) \preceq G_0$, whenever
  - $\pi^{-1}(1) = V(G_0)$, and
  - G is obtainable from G_0 through a finite (possibly empty) succession of the following two operations:

    (D_1) subdivide an arbitrary edge (thus creating a new 2-vertex);

    (D_2) identify any number of newly created 2-vertices.
- $\chi'_p(G, \pi) \leq \chi'_o(G_0)$

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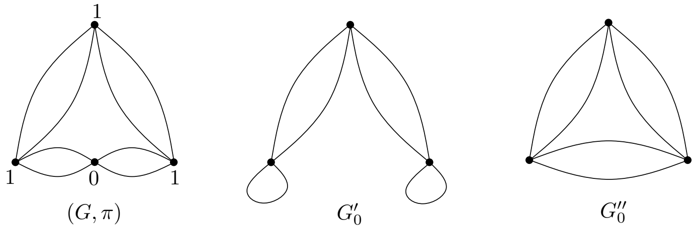
Theorem 4 (BL, Petruševski, Škrekovski [4])

For every connected proper parity pair (G, π) it holds $\chi'_p(G, \pi) \leq 6$. Furthermore, there exists a (not necessarily connected) graph G_0 such that $(G, \pi) \preceq G_0$ and $\chi'_p(G, \pi) = \chi'_o(G_0)$.

Derivative example



- The pair (G, π) is a derivative of both graphs G'_0 and G''_0. However, it holds that $\chi'_p(G, \pi) = 4$, whereas $\chi'_o(G'_0) = 4$ and $\chi'_o(G''_0) = 6$.

Characterization

Corollary 5 (BL, Petruševski, Škrekovski [4])

For every connected proper pair (G,π) that is not a derivative of Shannon's triangle of type $(2,2,2)$ or $(2,2,1)$, it holds that $\chi'_p(G,\pi) \le 4$.

- Hence, if π assigns 1 to at least four vertices of G, the vertex-parity chromatic index is at most 4.

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- Is this generalization useful?

Locally irregular edge-coloring

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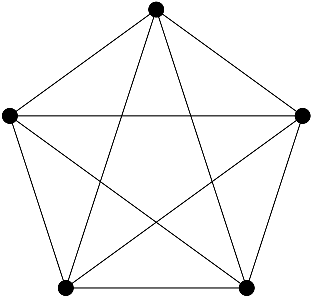
- Always improper—paths of odd length do not admit such a coloring
- Introduced by Baudon, Bensmail, Przybyło, and Woźniak in 2013 [1].
- Motivated by the (1-2-3)-conjecture:

  For every graph with no K_2 component there exists an edge weighting with 1, 2, and 3 such that for every two adjacent vertices the sums on their incident edges are distinct.

Example



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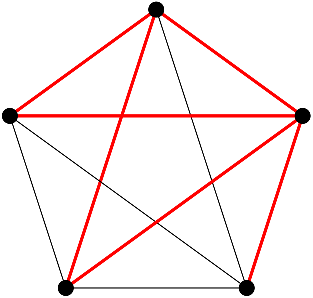

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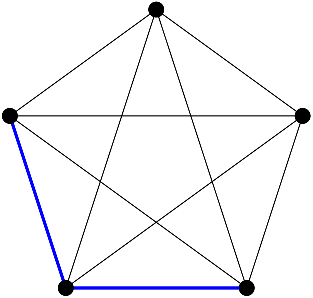

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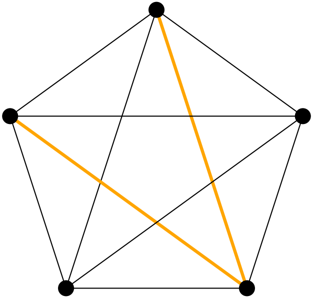

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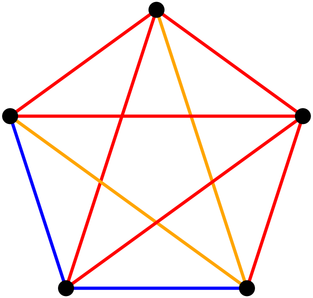

Decomposable graphs

- A graph is decomposable if it admits a locally irregular edge-coloring (LIE-C).
- The minimum k for which there is a LIE-C of a graph G with k colors is the locally irregular chromatic index of G, $\chi'_{\mathrm{irr}}(G)$.

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- The minimum k for which there is a LIE-C of a graph G with k colors is the locally irregular chromatic index of G, $\chi'_{\mathrm{irr}}(G)$.
- Not all graphs are decomposable, e.g. odd-length paths, odd-length cycles.
- A complete characterization was given by Baudon, Bensmail, Przybyło, and Woźniak [1].

The Conjecture

Conjecture 6 (Baudon et al., 2015)

For every decomposable graph G, it holds $\chi'_{\mathrm{irr}}(G) \leq 3$.

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- The conjecture, if true, is tight; consider e.g. C_6;
- It holds for graphs with large minimum degree [8] and regular graphs of large degree [1];
- In general, only constant upper bunds are known;

Results

Theorem 7 (BL, Przybyło, Soták [5])

For every decomposable graph G with maximum degree 3, it holds $\chi'_{\mathrm{irr}}(G) \leq 4$.

Theorem 8 (BL, Przybyło, Soták [5])

For every decomposable graph G, it holds $\chi'_{\mathrm{irr}}(G) \leq 220$.

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- A decomposable bipartite graph is balanced if all the vertices in one of the two partition parts have even degrees;

Vertex-parity edge-coloring application

- A decomposable bipartite graph is balanced if all the vertices in one of the two partition parts have even degrees;
- A trivial corollary of Theorem 4:

Theorem 9 (BL, Przybyło, Soták, 2016+)

Let G be a balanced graph. Then

$$\chi'_{\mathrm{irr}}(G) \leq 4\,.$$

- This observation decreases the previous upper bound of 328, due to Bensmail, Merker and Thomassen [2] to our 220 colors.

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Thank you!