On injective colorings of graphs (a survey)

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Injective colorings

- Introduced by Hahn, Kratochvíl, Širáň, and Sotteau in 2002
- Coloring of vertices such that every two vertices with a common neighbor receive distinct colors
- $\chi_i(G)$ – the injective chromatic number of a graph G
- Not necessarily proper
- Why injective? Its restriction to the (open) neighborhood of every vertex is injective
- Motivation from complexity and coding theory - injective coloring of hypercubes
- Related to distance colorings ($L(p,q)$-labelings)

Common neighbor graph

- *Common neighbor graph* $G^{(2)}$ of a graph G is a graph with the vertex set $V(G)$ and has an edge between two vertices if they have a common neighbor in G.
- $\chi_i(G) = \chi(G^{(2)})$
- Also *common neighborhood graph* or *open neighborhood graph*

Basic observations

Proposition 1 (Hahn et al.)

Let G be connected and distinct from K_2. Then $\chi_i(G) \geq \chi(G)$.

Proposition 2 (Hahn et al.)

Let G be a diameter 2 graph with independence number α. Then $\chi_i(G) \geq \alpha$.

Proposition 3 (Hahn et al.)

Let G be an arbitrary graph of order at least four. Then $\chi_i(G) = |V(G)|$ if and only if either G is a complete graph, or G has diameter 2 and every edge of G is contained in a triangle.

Proposition 4 (Hahn et al.)

Let G have maximum degree Δ. Then $\chi_i(G) \leq \Delta(\Delta - 1) + 1$.

Theorem 5 (Hahn et al.; Tarsi et al.)

Let Q_n be the n-dimensional cube. Then $\chi_i(Q_n) = n$ if and only if $n = 2^r$ for some $r \geq 0$.

Corollary 6 (Hahn et al.)

For any n, $\chi_i(Q_n) \leq 2^{\lceil \log n \rceil}$; thus, $\chi_i(Q_n) \leq 2n - 2$.

Theorem 7 (Hahn et al.)

$\chi_i(Q_{2^m - j}) = 2^m$ *for* $0 \leq j \leq 3$.

Open for other values of j

Theorem 8 (Hahn et al.)

Determining if there is an injective k-coloring of a graph G is NP-complete.

Jin, Xu and Zhang: determining whether there is an injective k-coloring of a balanced bipartite graph is NP-complete.

Theorem 9 (Huang, Lih)

For any graph G of order $n \geq 5$, the following statements hold.

1. *If $n = 5$ or even, then $n \leq \chi_i(G) + \chi_i(\overline{G}) \leq 2n$;*
2. *If $n \geq 7$ is odd, then $n + 1 \leq \chi_i(G) + \chi_i(\overline{G}) \leq 2n$.*

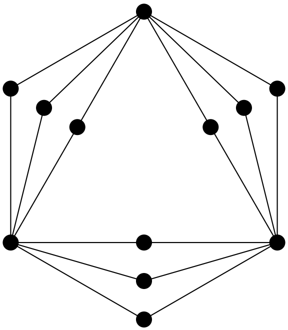
Theorem 10 (Huang, Lih)

For any graph G of order $n \geq 5$, $n \leq \chi_i(G)\chi_i(\overline{G}) \leq n^2$.

Theorem 11 (Chen et al.)

Let G be a K_4-minor free graph with $\Delta \geq 1$. Then,

$$\chi_i(G) \leq \left\lceil \frac{3}{2}\Delta \right\rceil .$$



Theorem 12 (Chen et al.)

If G is a planar graph with $\Delta \geq 3$, then $\chi_i(G) \leq \Delta^2 - \Delta$.

Conjecture 13 (Chen et al.)

For each planar graph G, $\chi_i(G) \leq \lceil \frac{3}{2}\Delta \rceil$.

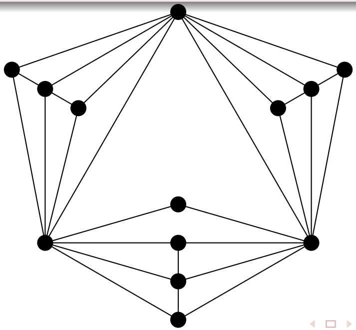
Theorem 14 (L., Škrekovski)

There exist planar graphs G of maximum degree $\Delta \geq 3$ satisfying the following:

(a) $\chi_i(G) = 5$, *if* $\Delta = 3$;

(b) $\chi_i(G) = \Delta + 5$, *if* $4 \leq \Delta \leq 7$;

(c) $\chi_i(G) = \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$, *if* $\Delta \geq 8$.



Conjecture 15 (L., Škrekovski)

Let G be a planar graph with maximum degree Δ. Then

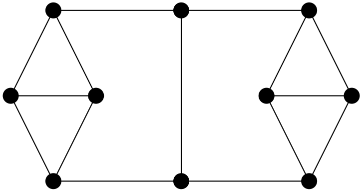
(a) $\chi_i(G) \leq 5$, *if* $\Delta = 3$;

(b) $\chi_i(G) \leq \Delta + 5$, *if* $4 \leq \Delta \leq 7$;

(c) $\chi_i(G) \leq \left\lfloor \frac{3}{2}\Delta \right\rfloor + 1$, *if* $\Delta \geq 8$.

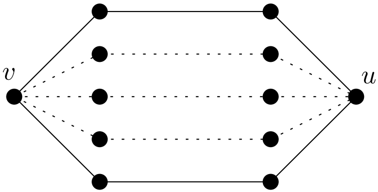
Subcubic planar graphs

- Subcubic planar graphs need at most 6 colors
- Conjecture says 5
- Computer check - up to 20 vertices the conjecture is true



Upper bound & girth condition

- Many papers on the topic
- List version introduced
- Planar graphs of girth 6, arbitrary $\Delta(G)$ and $\chi_i(G) = \Delta(G) + 1$.



$$d(u) = d(v) = \Delta(G)$$

		$g(G)$								
		5	6	7	8	9	10–11	12	13–18	19^+
$\Delta(G)$	3	$\Delta+3$	$\Delta+3$	$\Delta+2$	$\Delta+2$	$\Delta+2$	$\Delta+1$	$\Delta+1$	$\Delta+1$	Δ
	4	$\Delta+6$	$\Delta+3$	$\Delta+2$	$\Delta+2$	$\Delta+1$	$\Delta+1$	$\Delta+1$	Δ	Δ
	5	$\Delta+6$	$\Delta+3$	$\Delta+2$	$\Delta+1$	$\Delta+1$	$\Delta+1$	Δ	Δ	Δ
	6–7	$\Delta+6$	$\Delta+3$	$\Delta+2$	$\Delta+1$	$\Delta+1$	Δ	Δ	Δ	Δ
	8–9	$\Delta+6$	$\Delta+2$	$\Delta+2$	$\Delta+1$	$\Delta+1$	Δ	Δ	Δ	Δ
	10–12	$\Delta+6$	$\Delta+2$	$\Delta+2$	Δ	Δ	Δ	Δ	Δ	Δ
	13–15	$\Delta+4$	$\Delta+2$	$\Delta+2$	Δ	Δ	Δ	Δ	Δ	Δ
	16	$\Delta+4$	$\Delta+2$	Δ	Δ	Δ	Δ	Δ	Δ	Δ
	17–34	$\Delta+4$	$\Delta+1$	Δ	Δ	Δ	Δ	Δ	Δ	Δ
	35^+	$\Delta+3$	$\Delta+1$	Δ	Δ	Δ	Δ	Δ	Δ	Δ

Problem 16

Does there exist an integer M such that every planar graph G with maximum degree $\Delta(G) \geq M$ and girth at least 5 is injectively $(\Delta+1)$-colorable?

Questions

Thank you!