Edge-coloring Variations

Borut Lužar

Faculty of Information Studies, Novo mesto, Slovenia
borut.luzar@gmail.com
http://luzar.fis.unm.si

II Studencka Konferencja KNMD
May 27, 2017

Basics

- For a graph $G = (V, E)$, a *k-edge-coloring* is a function

$$f \; : \; E \mapsto \{1, 2, \ldots, k\}$$

(we can think of the k values as colors...)

Basics

- For a graph $G = (V, E)$, a *k-edge-coloring* is a function

$$f \ : \ E \mapsto \{1, 2, \ldots, k\}$$

  (we can think of the k values as colors...)
- Today, we consider only proper edge-colorings, i.e. adjacent edges receive distinct colors

Basics

- For a graph $G = (V, E)$, a *k-edge-coloring* is a function

$$f \; : \; E \mapsto \{1, 2, \ldots, k\}$$

(we can think of the k values as colors...)

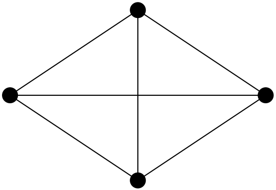
- Today, we consider only proper edge-colorings, i.e. adjacent edges receive distinct colors
- The smallest integer k for which G admits a (proper) k-edge-coloring is called the *chromatic index* of G, $\chi'(G)$

Basics

- For a graph $G = (V, E)$, a *k-edge-coloring* is a function

$$f : E \mapsto \{1, 2, \ldots, k\}$$

  (we can think of the k values as colors...)
- Today, we consider only proper edge-colorings, i.e. adjacent edges receive distinct colors
- The smallest integer k for which G admits a (proper) k-edge-coloring is called the *chromatic index* of G, $\chi'(G)$

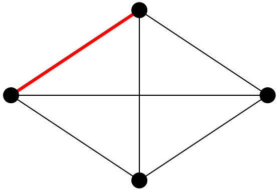


Basics

- For a graph $G = (V, E)$, a *k-edge-coloring* is a function

$$f \ : \ E \mapsto \{1, 2, \ldots, k\}$$

  (we can think of the k values as colors...)
- Today, we consider only proper edge-colorings, i.e. adjacent edges receive distinct colors
- The smallest integer k for which G admits a (proper) k-edge-coloring is called the *chromatic index* of G, $\chi'(G)$

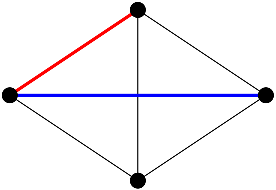


Basics

- For a graph $G = (V, E)$, a *k-edge-coloring* is a function

$$f \ : \ E \mapsto \{1, 2, \dots, k\}$$

  (we can think of the k values as colors...)
- Today, we consider only proper edge-colorings, i.e. adjacent edges receive distinct colors
- The smallest integer k for which G admits a (proper) k-edge-coloring is called the *chromatic index* of G, $\chi'(G)$

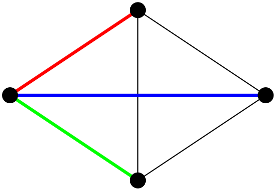


Basics

- For a graph $G = (V, E)$, a *k-edge-coloring* is a function

$$f \ : \ E \mapsto \{1, 2, \ldots, k\}$$

  (we can think of the k values as colors...)
- Today, we consider only proper edge-colorings, i.e. adjacent edges receive distinct colors
- The smallest integer k for which G admits a (proper) k-edge-coloring is called the *chromatic index* of G, $\chi'(G)$

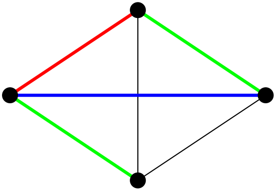


Basics

- For a graph $G = (V, E)$, a *k-edge-coloring* is a function

$$f \ : \ E \mapsto \{1, 2, \ldots, k\}$$

  (we can think of the k values as colors...)
- Today, we consider only proper edge-colorings, i.e. adjacent edges receive distinct colors
- The smallest integer k for which G admits a (proper) k-edge-coloring is called the *chromatic index* of G, $\chi'(G)$

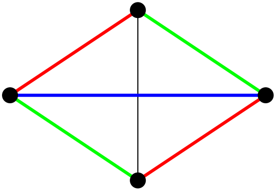


Basics

- For a graph $G = (V, E)$, a *k-edge-coloring* is a function

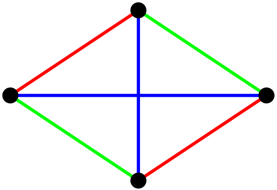
$$f \; : \; E \mapsto \{1, 2, \ldots, k\}$$

  (we can think of the k values as colors...)
- Today, we consider only proper edge-colorings, i.e. adjacent edges receive distinct colors
- The smallest integer k for which G admits a (proper) k-edge-coloring is called the *chromatic index* of G, $\chi'(G)$



Basics

- For a graph $G = (V, E)$, a *k-edge-coloring* is a function
$$f \ : \ E \mapsto \{1, 2, \dots, k\}$$
(we can think of the k values as colors...)
- Today, we consider only proper edge-colorings, i.e. adjacent edges receive distinct colors
- The smallest integer k for which G admits a (proper) k-edge-coloring is called the *chromatic index* of G, $\chi'(G)$



$\longrightarrow \quad \chi'(K_4) = 3$

Vizing's Theorem

- Our main goal today: determine chromatic index as accurate as possible (for a selected class of graphs)!

Vizing's Theorem

- Our main goal today: determine chromatic index as accurate as possible (for a selected class of graphs)!
- Lower bound: adjacent edges must receive distinct colors, so

$$\Delta(G) \leq \chi'(G)$$

Vizing's Theorem

- Our main goal today: determine chromatic index as accurate as possible (for a selected class of graphs)!
- Lower bound: adjacent edges must receive distinct colors, so

$$\Delta(G) \leq \chi'(G)$$

Theorem 1 (Vizing, 1964)

For every (simple) graph G

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Planar graphs

Conjecture 2 (Vizing, 1965)

For every planar graph G with $\Delta(G) \geq 6$

$$\chi'(G) = \Delta(G).$$

Planar graphs

Conjecture 2 (Vizing, 1965)

For every planar graph G with $\Delta(G) \geq 6$

$$\chi'(G) = \Delta(G).$$

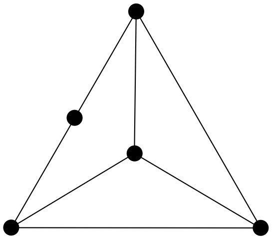
Theorem 3 (Sanders & Zhao, 2001)

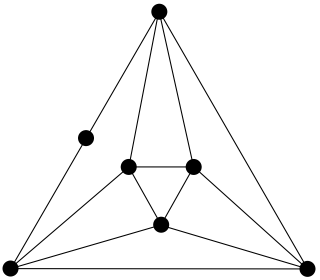
For every planar graph G with $\Delta(G) \geq 7$

$$\chi'(G) = \Delta(G).$$

Planar graphs

- For planar graphs of maximum degree at most 5, there are graphs G that need $\Delta(G)+1$ colors, e.g.:





Bipartite and complete graphs

Theorem 4 (König, 1916)

For every bipartite graph G

$$\chi'(G) = \Delta(G).$$

Bipartite and complete graphs

Theorem 4 (König, 1916)

For every bipartite graph G

$$\chi'(G) = \Delta(G).$$

- For complete graphs K_{2k}, we have $2k-1$ disjoint perfect matchings; we assign the same color to all edges of a matching, so:

$$\chi'(K_{2k}) = \Delta(K_{2k}) = 2k-1.$$

Complete graphs of odd order need additional color:

$$\chi'(K_{2k+1}) = \Delta(K_{2k+1}) + 1 = 2k+1.$$

Adding assumptions

- How do the bounds for chromatic index change if we add additional assumptions to the coloring?

Adding assumptions

- How do the bounds for chromatic index change if we add additional assumptions to the coloring?
- We will focus on three types:
  - Acyclic edge-coloring;
  - Strong edge-coloring;
  - Star edge-coloring.

Acyclic edge-coloring

Acyclic edge-coloring

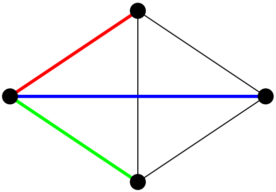
- An *acyclic k-edge-coloring* of a graph G is a k-edge-coloring where the edges of every cycle are assigned at least three distinct colors, i.e., there are no bichromatic cycles.

Acyclic edge-coloring

- An *acyclic k-edge-coloring* of a graph G is a k-edge-coloring where the edges of every cycle are assigned at least three distinct colors, i.e., there are no bichromatic cycles.
- The smallest k for which an acyclic k-edge coloring of G exists is the *acyclic chromatic index* of G, $\chi'_a(G)$.

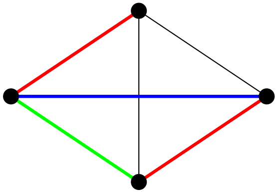
Acyclic edge-coloring

- An *acyclic k-edge-coloring* of a graph G is a k-edge-coloring where the edges of every cycle are assigned at least three distinct colors, i.e., there are no bichromatic cycles.
- The smallest k for which an acyclic k-edge coloring of G exists is the *acyclic chromatic index* of G, $\chi'_a(G)$.



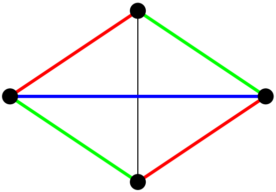
Acyclic edge-coloring

- An *acyclic k-edge-coloring* of a graph G is a k-edge-coloring where the edges of every cycle are assigned at least three distinct colors, i.e., there are no bichromatic cycles.
- The smallest k for which an acyclic k-edge coloring of G exists is the *acyclic chromatic index* of G, $\chi'_a(G)$.



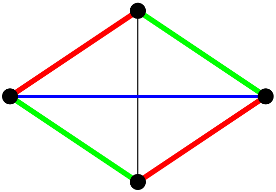
Acyclic edge-coloring

- An *acyclic k-edge-coloring* of a graph G is a k-edge-coloring where the edges of every cycle are assigned at least three distinct colors, i.e., there are no bichromatic cycles.
- The smallest k for which an acyclic k-edge coloring of G exists is the *acyclic chromatic index* of G, $\chi'_a(G)$.



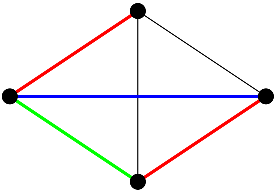
Acyclic edge-coloring

- An *acyclic k-edge-coloring* of a graph G is a k-edge-coloring where the edges of every cycle are assigned at least three distinct colors, i.e., there are no bichromatic cycles.
- The smallest k for which an acyclic k-edge coloring of G exists is the *acyclic chromatic index* of G, $\chi'_a(G)$.



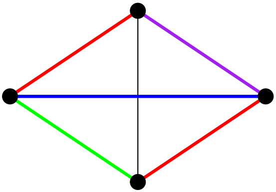
Acyclic edge-coloring

- An *acyclic k-edge-coloring* of a graph G is a k-edge-coloring where the edges of every cycle are assigned at least three distinct colors, i.e., there are no bichromatic cycles.
- The smallest k for which an acyclic k-edge coloring of G exists is the *acyclic chromatic index* of G, $\chi'_a(G)$.



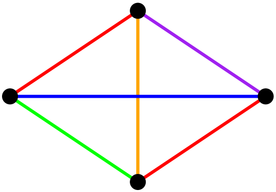
Acyclic edge-coloring

- An *acyclic k-edge-coloring* of a graph G is a k-edge-coloring where the edges of every cycle are assigned at least three distinct colors, i.e., there are no bichromatic cycles.
- The smallest k for which an acyclic k-edge coloring of G exists is the *acyclic chromatic index* of G, $\chi'_a(G)$.



Acyclic edge-coloring

- An *acyclic k-edge-coloring* of a graph G is a k-edge-coloring where the edges of every cycle are assigned at least three distinct colors, i.e., there are no bichromatic cycles.
- The smallest k for which an acyclic k-edge coloring of G exists is the *acyclic chromatic index* of G, $\chi'_a(G)$.



The Conjecture

Conjecture 5 (Fiamčík, 1978; Alon, Sudakov, Zaks, 2001)

For every graph G it holds

$$\Delta(G) \leq \chi'_a(G) \leq \Delta(G) + 2.$$

The Conjecture

Conjecture 5 (Fiamčík, 1978; Alon, Sudakov, Zaks, 2001)

For every graph G it holds

$$\Delta(G) \leq \chi'_a(G) \leq \Delta(G) + 2.$$

- Super-surprising: Only one additional color is enough?!

The Conjecture

Conjecture 5 (Fiamčík, 1978; Alon, Sudakov, Zaks, 2001)

For every graph G it holds

$$\Delta(G) \leq \chi'_a(G) \leq \Delta(G) + 2.$$

- Super-surprising: Only one additional color is enough?!
- If G is d-regular, then $\chi'_a(G) \geq d + 1$.

The Conjecture

Conjecture 5 (Fiamčík, 1978; Alon, Sudakov, Zaks, 2001)

For every graph G it holds

$$\Delta(G) \leq \chi'_a(G) \leq \Delta(G) + 2.$$

- Super-surprising: Only one additional color is enough?!
- If G is d-regular, then $\chi'_a(G) \geq d + 1$.

  Otherwise, every pair of colors induces a 2-factor, since every color is present at every vertex.

The Conjecture

Conjecture 5 (Fiamčík, 1978; Alon, Sudakov, Zaks, 2001)

For every graph G it holds

$$\Delta(G) \leq \chi'_a(G) \leq \Delta(G) + 2.$$

- Super-surprising: Only one additional color is enough?!
- If G is d-regular, then $\chi'_a(G) \geq d + 1$.

  Otherwise, every pair of colors induces a 2-factor, since every color is present at every vertex.
- Super-surprising 2: Conjecture 5 is not confirmed even for complete graphs!

Complete graphs

Proposition 6

For even complete graph K_{2n} and $F \subset E(K_{2n})$, $|F| \leq n-2$, it holds

$$\chi'_a(K_{2n} \backslash F) \geq 2n+1 = \Delta(K_{2n} \backslash F) + 2\,.$$

Complete graphs

Proposition 6

For even complete graph K_{2n} and $F \subset E(K_{2n})$, $|F| \leq n-2$, it holds

$$\chi'_a(K_{2n} \backslash F) \geq 2n+1 = \Delta(K_{2n} \backslash F) + 2\,.$$

Proof.

- There are at most n edges of one color and they induce a perfect matching. Hence, the other colors are assigned to at most $n-1$ edges.

Complete graphs

Proposition 6

For even complete graph K_{2n} and $F \subset E(K_{2n})$, $|F| \leq n-2$, it holds

$$\chi'_a(K_{2n} \backslash F) \geq 2n+1 = \Delta(K_{2n} \backslash F) + 2\,.$$

Proof.

- There are at most n edges of one color and they induce a perfect matching. Hence, the other colors are assigned to at most $n-1$ edges.
- There are $2n^2 - n$ edges in K_{2n}.

Complete graphs

Proposition 6

For even complete graph K_{2n} and $F \subset E(K_{2n})$, $|F| \leq n-2$, it holds

$$\chi'_a(K_{2n} \backslash F) \geq 2n+1 = \Delta(K_{2n} \backslash F) + 2\,.$$

Proof.

- There are at most n edges of one color and they induce a perfect matching. Hence, the other colors are assigned to at most $n-1$ edges.
- There are $2n^2 - n$ edges in K_{2n}.
- Using only $2n = \Delta(K_{2n}) + 1$ colors leaves at least $n-1$ edges uncolored.

Perfect 1-factorization

Conjecture 7 (Kotzig, 1964)

For every $n \geq 2$, K_{2n} can be decomposed into $2n-1$ perfect matchings such that the union of any two matchings forms a hamiltonian cycle in K_{2n}.

- Closely related to acyclic edge-colorings.
- If the Conjecture 7 is true, the removal of one vertex from K_{2n} results in an acyclic edge coloring of K_{2n-1} with $2n-1 = \Delta(K_{2n-1}) + 1$ colors, which is optimal.

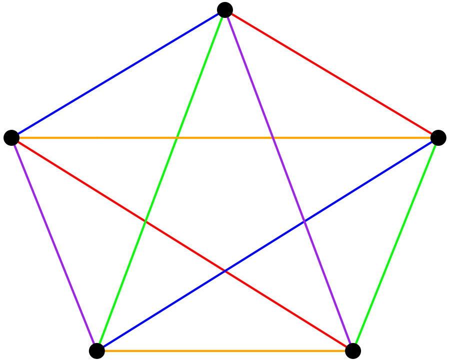
Perfect 1-factorization

Conjecture 7 (Kotzig, 1964)

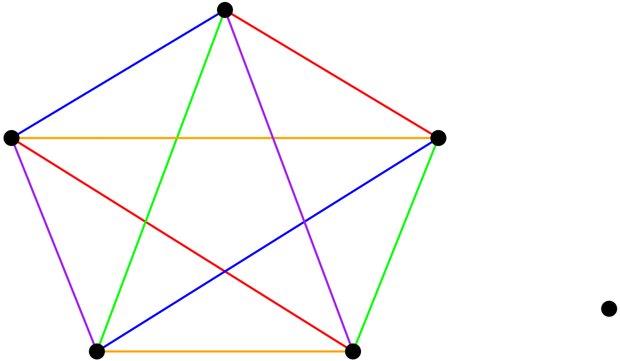
For every $n \geq 2$, K_{2n} can be decomposed into $2n-1$ perfect matchings such that the union of any two matchings forms a hamiltonian cycle in K_{2n}.

- Closely related to acyclic edge-colorings.
- If the Conjecture 7 is true, the removal of one vertex from K_{2n} results in an acyclic edge coloring of K_{2n-1} with $2n-1 = \Delta(K_{2n-1}) + 1$ colors, which is optimal.
- If K_{n+1} has perfect 1-factorization, then $K_{n,n}$ has it also.

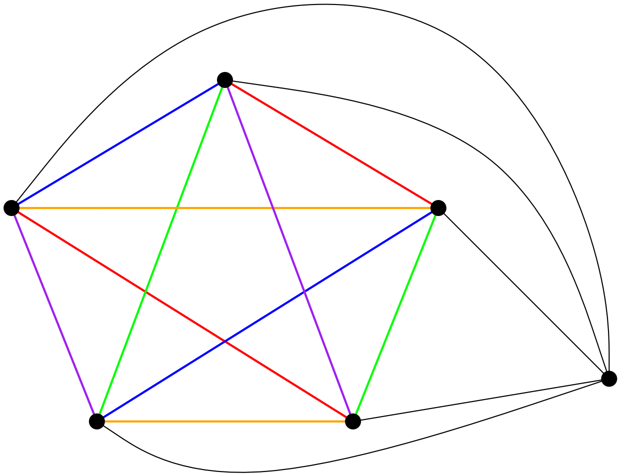
Perfect 1-factorization



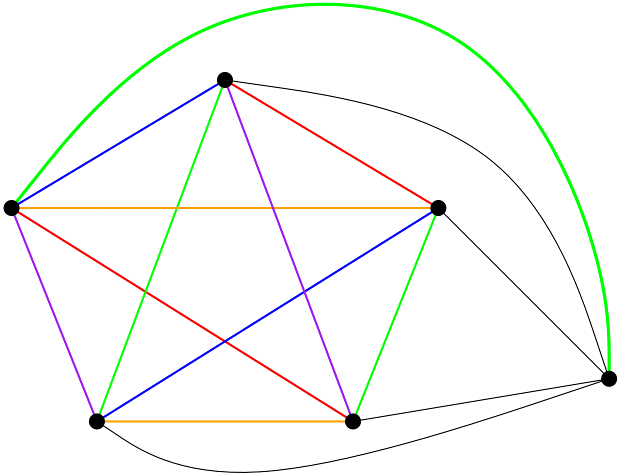
Perfect 1-factorization



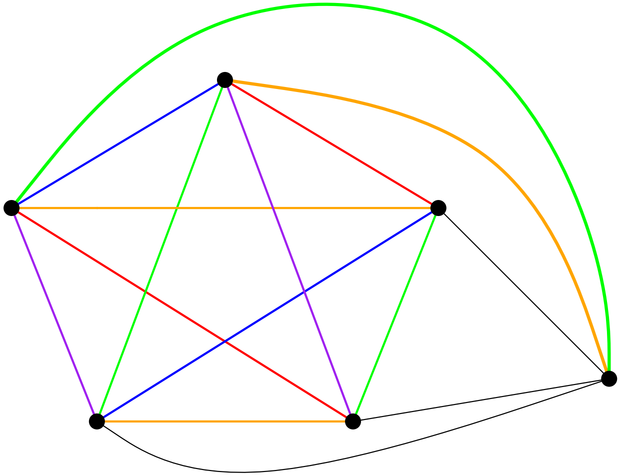
Perfect 1-factorization



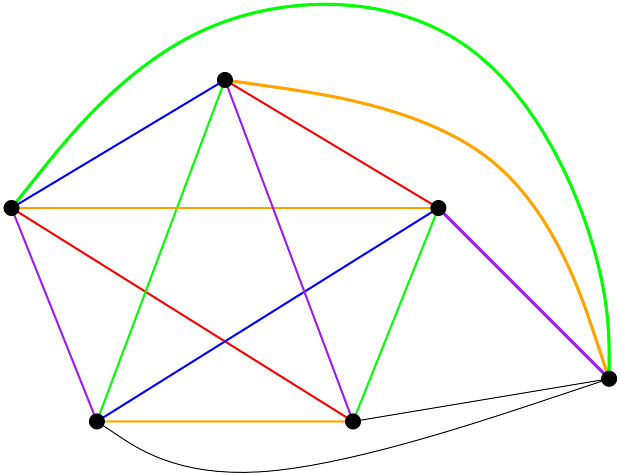
Perfect 1-factorization



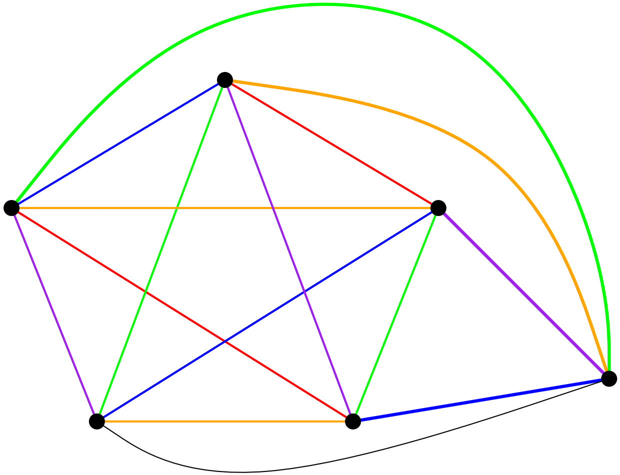
Perfect 1-factorization



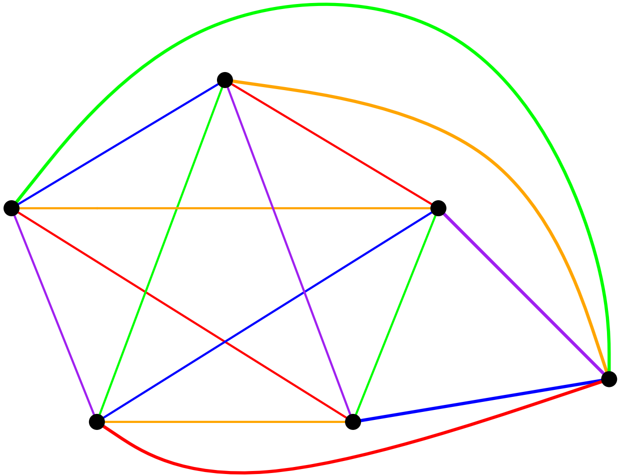
Perfect 1-factorization



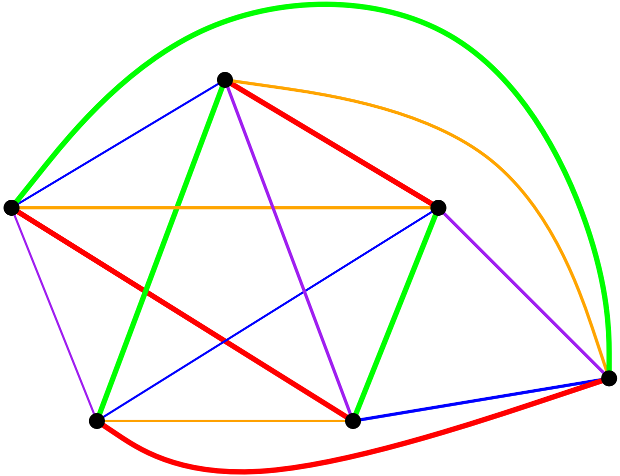
Perfect 1-factorization



Perfect 1-factorization



Perfect 1-factorization



General graphs

- Several upper bounds have been proven repeatedly, all using probabilistic approaches

General graphs

- Several upper bounds have been proven repeatedly, all using probabilistic approaches

Theorem 8 (Giotis et al., 2017)

For every graph G it holds

$$\chi'_a(G) \leq \lceil 3.74\ (\Delta(G) - 1) \rceil + 1$$

General graphs

- Several upper bounds have been proven repeatedly, all using probabilistic approaches

Theorem 8 (Giotis et al., 2017)

For every graph G it holds

$$\chi'_a(G) \leq \lceil 3.74\ (\Delta(G) - 1) \rceil + 1$$

Theorem 9 (Alon, Sudakov, Zaks, 2001)

For every graph G with girth at least $C\Delta(G) \log \Delta(G)$, for a constant C, it holds

$$\chi'_a(G) \leq \Delta(G) + 2\,.$$

Subcubic graphs

- The notion of acyclic colorings was first introduced in 1973 by Grűnbaum for the vertex version. In 1979, Burnstein proved that 5 colors suffice for acyclic vertex coloring of every graph G with $\Delta(G) \leq 4$.

Subcubic graphs

- The notion of acyclic colorings was first introduced in 1973 by Grűnbaum for the vertex version. In 1979, Burnstein proved that 5 colors suffice for acyclic vertex coloring of every graph G with $\Delta(G) \leq 4$.
- The maximum degree of a line graph $L(G)$ of a subcubic graph G is at most 4...

Subcubic graphs

- The notion of acyclic colorings was first introduced in 1973 by Grűnbaum for the vertex version. In 1979, Burnstein proved that 5 colors suffice for acyclic vertex coloring of every graph G with $\Delta(G) \leq 4$.
- The maximum degree of a line graph $L(G)$ of a subcubic graph G is at most 4...

Corollary 10 (Burnstein, 1979)

Let G be a subcubic graph. Then

$$\chi'_a(G) \leq 5\,.$$

Hypercubes

- Muthu considered acyclic edge-coloring of Cartesian products of graphs. As a corollary, he obtained a bound for hypercubes

Hypercubes

- Muthu considered acyclic edge-coloring of Cartesian products of graphs. As a corollary, he obtained a bound for hypercubes

Theorem 11 (Muthu, 2007)

For hypercubes Q_n of dimension $n \geq 2$ it holds

$$\chi'_a(Q_n) = n + 1 \, .$$

Planar graphs

- Another candidate class of graphs to confirm Conjecture 7

Planar graphs

- Another candidate class of graphs to confirm Conjecture 7
- Confirmed for triangle-free planar graphs

Planar graphs

- Another candidate class of graphs to confirm Conjecture 7
- Confirmed for triangle-free planar graphs
- In general we are close...

Planar graphs

- Another candidate class of graphs to confirm Conjecture 7
- Confirmed for triangle-free planar graphs
- In general we are close...

Theorem 12 (Wang, Zhang, 2017+)

Let G be a planar graph. Then

$$\chi'_a(G) \leq \Delta(G) + 6\,.$$

Planar graphs - Δ colors

- Cohen, Havet and Müller conjectured that all planar graphs with sufficiently large maximum degree have acyclic chromatic index equal to Δ

Planar graphs - Δ colors

- Cohen, Havet and Müller conjectured that all planar graphs with sufficiently large maximum degree have acyclic chromatic index equal to Δ

Theorem 13 (Cranston, 2017+)

Let G be a planar graph with $\Delta(G) \geq 4.2 \cdot 10^{14}$. Then

$$\chi'_a(G) = \Delta(G) \, .$$

Strong edge-coloring

Definition

- **Distance between edges**: distance between corresponding vertices in the line graph (adjacent edges are at distance 1)

Definition

- **Distance between edges**: distance between corresponding vertices in the line graph (adjacent edges are at distance 1)
- A *strong k-edge-coloring* of a graph G is a proper k-edge-coloring where the edges of every path of length 3 have three distinct colors, i.e., not only incident edges but also the edges at distance 2 have distinct colors.

Definition

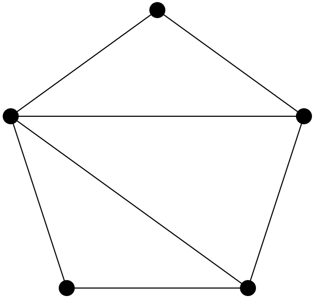
- **Distance between edges**: distance between corresponding vertices in the line graph (adjacent edges are at distance 1)
- A *strong k-edge-coloring* of a graph G is a proper k-edge-coloring where the edges of every path of length 3 have three distinct colors, i.e., not only incident edges but also the edges at distance 2 have distinct colors.
- The smallest k for which G admits a strong k-edge-coloring is the *strong chromatic index* of G, $\chi'_s(G)$.

Definition

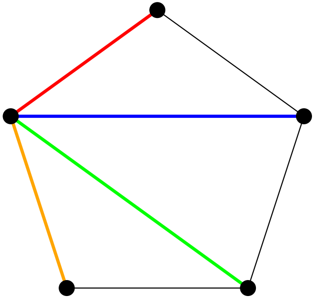
- **Distance between edges**: distance between corresponding vertices in the line graph (adjacent edges are at distance 1)
- A *strong k-edge-coloring* of a graph G is a proper k-edge-coloring where the edges of every path of length 3 have three distinct colors, i.e., not only incident edges but also the edges at distance 2 have distinct colors.
- The smallest k for which G admits a strong k-edge-coloring is the *strong chromatic index* of G, $\chi'_s(G)$.
- Strong edge coloring of G is a *vertex* 2*-distance coloring* of its line graph $L(G)$

$$\chi'_s(G) = \chi(L(G)^2) .$$

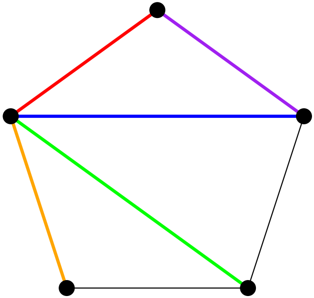
Example



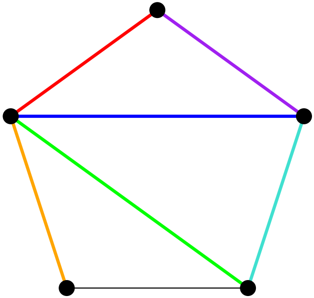
Example



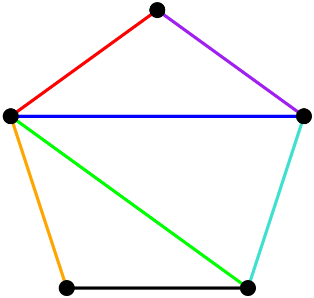
Example



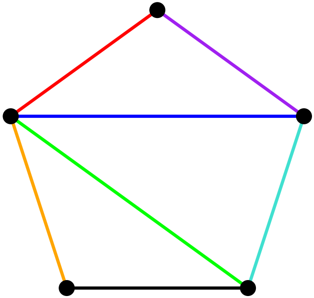
Example



Example



Example



$$\chi'_s(G) = 7$$

Quick test

- Determine $\chi'_s(K_n)$!

Quick test

- Determine $\chi'_s(K_n)$!
  Yes, $\chi'_s(K_n) = \binom{n}{2}$

Quick test

- Determine $\chi'_s(K_n)$!
  Yes, $\chi'_s(K_n) = \binom{n}{2}$
- Determine $\chi'_s(K_{n,m})$!

Quick test

- Determine $\chi'_s(K_n)$!
  Yes, $\chi'_s(K_n) = \binom{n}{2}$
- Determine $\chi'_s(K_{n,m})$!
  Yes, $\chi'_s(K_{n,m}) = n \cdot m$

Quick test

- Determine $\chi'_s(K_n)$!
  Yes, $\chi'_s(K_n) = \binom{n}{2}$
- Determine $\chi'_s(K_{n,m})$!
  Yes, $\chi'_s(K_{n,m}) = n \cdot m$
- Determine $\chi'_s(P_n)$!

Quick test

- Determine $\chi'_s(K_n)$!
  Yes, $\chi'_s(K_n) = \binom{n}{2}$
- Determine $\chi'_s(K_{n,m})$!
  Yes, $\chi'_s(K_{n,m}) = n \cdot m$
- Determine $\chi'_s(P_n)$!
  Yes, $\chi'_s(P_n) = 3$, if $n \geq 4$

Quick test

- Determine $\chi'_s(K_n)$!
  Yes, $\chi'_s(K_n) = \binom{n}{2}$
- Determine $\chi'_s(K_{n,m})$!
  Yes, $\chi'_s(K_{n,m}) = n \cdot m$
- Determine $\chi'_s(P_n)$!
  Yes, $\chi'_s(P_n) = 3$, if $n \geq 4$
- Determine $\chi'_s(C_n)$!

Quick test

- Determine $\chi'_s(K_n)$!
  Yes, $\chi'_s(K_n) = \binom{n}{2}$
- Determine $\chi'_s(K_{n,m})$!
  Yes, $\chi'_s(K_{n,m}) = n \cdot m$
- Determine $\chi'_s(P_n)$!
  Yes, $\chi'_s(P_n) = 3$, if $n \geq 4$
- Determine $\chi'_s(C_n)$!
  Yes, $3 \leq \chi'_s(C_n) \leq 5$

The Conjecture

- Strong edge-coloring was initiated by Fouquet and Jolivet in 1982.

The Conjecture

- Strong edge-coloring was initiated by Fouquet and Jolivet in 1982.
- Erdős and Nešetřil in 1985 proposed a conjecture on the upper bound.

The Conjecture

- Strong edge-coloring was initiated by Fouquet and Jolivet in 1982.
- Erdős and Nešetřil in 1985 proposed a conjecture on the upper bound.

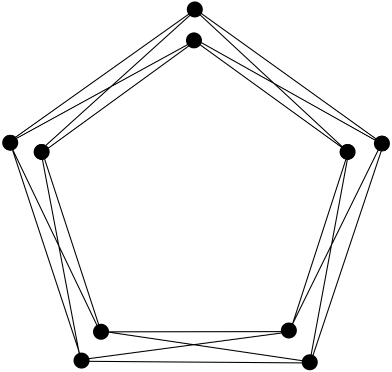
Conjecture 14 (Erdős, Nešetřil, 1985)

For every graph G it holds

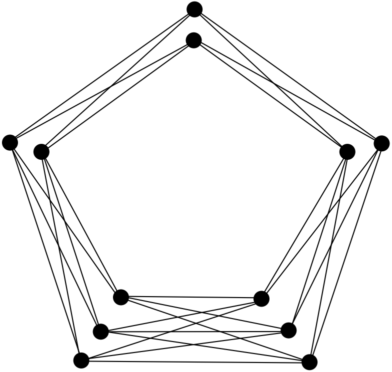
$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta(G)^2\,, & \Delta(G) \text{ is even;} \\ \frac{1}{4}(5\Delta(G)^2 - 2\Delta(G) + 1)\,, & \Delta(G) \text{ is odd.} \end{cases}$$

The Conjecture

- The bounds in Conjecture 14 are tight:



$\Delta(G) = 4$



$\Delta(G) = 5$

The Conjecture

The construction of graphs achieving the conjectured bound:

- For even Δ replace each vertex of a 5-cycle with $\frac{\Delta}{2}$ vertices;
- For odd Δ replace two consecutive vertices of a 5-cycle with $\frac{\Delta+1}{2}$ vertices and the others with $\frac{\Delta-1}{2}$ vertices.

General graphs

- By greedy method, we have $\chi'_s(G) \leq 2\Delta(G)(\Delta(G)-1)+1$

General graphs

- By greedy method, we have $\chi'_s(G) \leq 2\Delta(G)(\Delta(G) - 1) + 1$
- As in the acyclic case, several upper bounds have been proven repeatedly, all using probabilistic approaches

General graphs

- By greedy method, we have $\chi'_s(G) \leq 2\Delta(G)(\Delta(G)-1)+1$
- As in the acyclic case, several upper bounds have been proven repeatedly, all using probabilistic approaches

Theorem 15 (Bonamy, Perret, Postle, 2017+)

For every graph G with sufficiently large maximum degree it holds

$$\chi'_s(G) \leq 1.835\Delta(G)^2$$

Subcubic & Subquartic graphs

Theorem 16 (Andersen, 1992)

Let G be a graph with $\Delta(G) = 3$. Then,

$$\chi'_s(G) \leq 10\,.$$

Subcubic & Subquartic graphs

Theorem 16 (Andersen, 1992)

Let G be a graph with $\Delta(G) = 3$. Then,

$$\chi'_s(G) \leq 10 \, .$$

Theorem 17 (Cranston, 2006)

Let G be a graph with $\Delta(G) = 4$. Then,

$$\chi'_s(G) \leq 22 \, .$$

Bipartite graphs

Conjecture 18 (Faudree et al., 1990)

Let G be a bipartite graph. Then,

$$\chi'_s(G) \leq \Delta(G)^2 .$$

Bipartite graphs

Conjecture 18 (Faudree et al., 1990)

Let G be a bipartite graph. Then,

$$\chi'_s(G) \leq \Delta(G)^2 .$$

- And even stronger version:

Bipartite graphs

Conjecture 18 (Faudree et al., 1990)

Let G be a bipartite graph. Then,

$$\chi'_s(G) \leq \Delta(G)^2 \,.$$

- And even stronger version:

Conjecture 19 (Brualdi, Quinn Massey, 1993)

If G is bipartite graph with maximum degree of partite sets Δ_1 and Δ_2, then

$$\chi'_s(G) \leq \Delta_1 \cdot \Delta_2 \,.$$

Bipartite graphs

Theorem 20 (Steger, Yu, 1993)

Let G be a subcubic bipartite graph. Then,

$$\chi'_s(G) \leq 9\,.$$

Bipartite graphs

Theorem 20 (Steger, Yu, 1993)

Let G be a subcubic bipartite graph. Then,

$$\chi'_s(G) \leq 9\,.$$

Theorem 21 (Nakprasit, 2008)

Let G be a bipartite graph with maximum degree of partite sets 2 and Δ, then

$$\chi'_s(G) \leq 2\Delta\,.$$

Bipartite graphs

Theorem 20 (Steger, Yu, 1993)

Let G be a subcubic bipartite graph. Then,

$$\chi'_s(G) \leq 9\,.$$

Theorem 21 (Nakprasit, 2008)

Let G be a bipartite graph with maximum degree of partite sets 2 and Δ, then

$$\chi'_s(G) \leq 2\Delta\,.$$

- For $(3, \Delta)$-graphs there is a weaker result: $\chi'_s(G) \leq 4\Delta$

Hypercubes

Theorem 22 (Faudree et al., 1990)

For a d-dimensional hypercube Q_d it holds

$$\chi'_s(Q_d) = 2\,d \quad \textit{if } d \geq 2.$$

Hypercubes

Theorem 22 (Faudree et al., 1990)

For a d-dimensional hypercube Q_d it holds

$$\chi'_s(Q_d) = 2\,d \quad \textit{if } d \geq 2.$$

Proof.

- Lower bound is achieved by C_4 and edges incident to two of its consecutive vertices.

Hypercubes

Theorem 22 (Faudree et al., 1990)

For a d-dimensional hypercube Q_d it holds

$$\chi'_s(Q_d) = 2\,d \quad \textit{if } d \geq 2.$$

Proof.

- Lower bound is achieved by C_4 and edges incident to two of its consecutive vertices.
- For the upper bound, take the set of edges E_i of i-th dimension

Hypercubes

Theorem 22 (Faudree et al., 1990)

For a d-dimensional hypercube Q_d it holds

$$\chi'_s(Q_d) = 2\,d \quad \textit{if } d \geq 2.$$

Proof.

- Lower bound is achieved by C_4 and edges incident to two of its consecutive vertices.
- For the upper bound, take the set of edges E_i of i-th dimension
- Divide E_i into two sets E_i^0 and E_i^1, depending on the parity of the sum of all coordinates (except i-th) of one of endvertices (0 or 1, respectively)

Hypercubes

Theorem 22 (Faudree et al., 1990)

For a d-dimensional hypercube Q_d it holds

$$\chi'_s(Q_d) = 2\,d \quad \textit{if } d \geq 2.$$

Proof.

- Lower bound is achieved by C_4 and edges incident to two of its consecutive vertices.
- For the upper bound, take the set of edges E_i of i-th dimension
- Divide E_i into two sets E_i^0 and E_i^1, depending on the parity of the sum of all coordinates (except i-th) of one of endvertices (0 or 1, respectively)
- Color all E_i^0 and E_i^1 with distinct colors

Planar graphs

Theorem 23 (Faudree et al., 1990)

Let G be a planar graph. Then,

$$\chi'_s(G) \leq 4\,\Delta(G) + 4\,.$$

Theorem 23 (Faudree et al., 1990)

Let G be a planar graph. Then,

$$\chi'_s(G) \leq 4\,\Delta(G) + 4\,.$$

Proof.

- Color G properly with $\chi'(G)$ colors

Planar graphs

Theorem 23 (Faudree et al., 1990)

Let G be a planar graph. Then,

$$\chi'_s(G) \leq 4\,\Delta(G) + 4\,.$$

Proof.

- Color G properly with $\chi'(G)$ colors
- Let M_i be the set of the edges of same color. Let $G(M_i)$ be a graph induced by M_i where every edge from M_i is contracted

Planar graphs

Theorem 23 (Faudree et al., 1990)

Let G be a planar graph. Then,

$$\chi'_s(G) \leq 4\,\Delta(G) + 4\,.$$

Proof.

- Color G properly with $\chi'(G)$ colors
- Let M_i be the set of the edges of same color. Let $G(M_i)$ be a graph induced by M_i where every edge from M_i is contracted
- Since $G(M_i)$ is planar, its vertices (the edges of M_i) can be colored with 4 colors by the Four Color Theorem, hence all the edges with a common edge receive distinct color

Planar graphs

Theorem 23 (Faudree et al., 1990)

Let G be a planar graph. Then,

$$\chi'_s(G) \leq 4\,\Delta(G) + 4\,.$$

Proof.

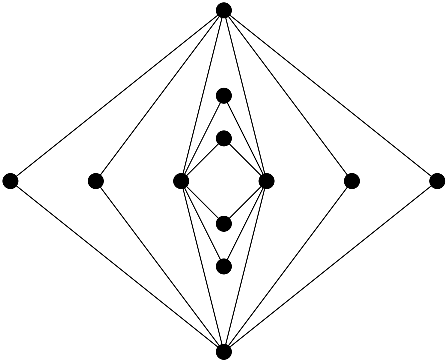
- Color G properly with $\chi'(G)$ colors
- Let M_i be the set of the edges of same color. Let $G(M_i)$ be a graph induced by M_i where every edge from M_i is contracted
- Since $G(M_i)$ is planar, its vertices (the edges of M_i) can be colored with 4 colors by the Four Color Theorem, hence all the edges with a common edge receive distinct color
- Altogether we need $4\,\chi'(G)$ colors

Planar graphs

- The above bound is pretty tight: Faudree et al. presented a construction of planar graphs with $\chi'_s(G) = 4\Delta(G) - 4$

Planar graphs

- The above bound is pretty tight: Faudree et al. presented a construction of planar graphs with $\chi'_s(G) = 4\Delta(G) - 4$
- Join two copies of $K_{2,m}$ along a fixed 4-cycle.



Planar graphs

- Forbidding short cycles in planar graphs, gives us some more freedom

Planar graphs

- Forbidding short cycles in planar graphs, gives us some more freedom

Conjecture 24 (Hudák et al., 2014)

There exists a constant C such that for every planar graph G with girth $k \geq 5$ it holds

$$\chi'_s(G) \leq \left\lceil \frac{2k(\Delta(G)-1)}{k-1} \right\rceil + C$$

Star edge-coloring

Definition

- Somewhere between strong edge-coloring and acyclic edge-coloring

Definition

- Somewhere between strong edge-coloring and acyclic edge-coloring
- Forbidding longer bichromatic paths as strong and only short bichromatic cycles

Definition

- Somewhere between strong edge-coloring and acyclic edge-coloring
- Forbidding longer bichromatic paths as strong and only short bichromatic cycles
- A *star edge-coloring* of a graph G is a proper edge-coloring without bichromatic paths and cycles of length 4

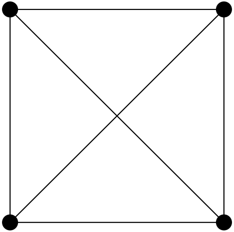
Definition

- Somewhere between strong edge-coloring and acyclic edge-coloring
- Forbidding longer bichromatic paths as strong and only short bichromatic cycles
- A *star edge-coloring* of a graph G is a proper edge-coloring without bichromatic paths and cycles of length 4
- The smallest k for which a star k-edge-coloring of G exists is the *star chromatic index* of G, $\chi'_{\mathrm{st}}(G)$.

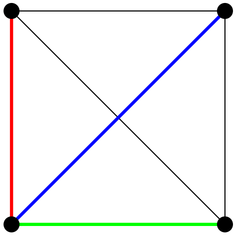
Definition

- Somewhere between strong edge-coloring and acyclic edge-coloring
- Forbidding longer bichromatic paths as strong and only short bichromatic cycles
- A *star edge-coloring* of a graph G is a proper edge-coloring without bichromatic paths and cycles of length 4
- The smallest k for which a star k-edge-coloring of G exists is the *star chromatic index* of G, $\chi'_{\mathrm{st}}(G)$.
- The name "star" comes from the vertex version where every pair of colors induces a star forest

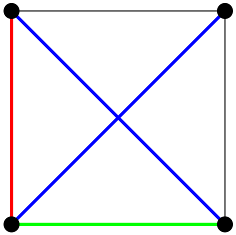
Example



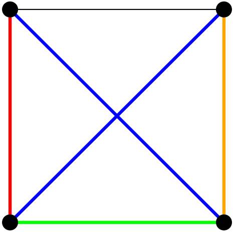
Example



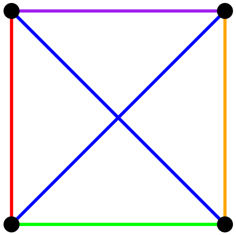
Example



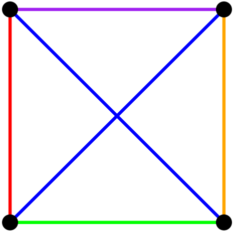
Example



Example



Example



$\chi'_{\mathrm{st}}(K_4) = 5$

Complete graphs

Theorem 25 (Dvořák, Mohar, Šámal, 2013)

The star chromatic index of the complete graph K_n satisfies

$$2n\frac{n-1}{n+2} \leq \chi'_{\mathrm{st}}(K_n) \leq n\frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{1/4}}.$$

In particular, for every $\epsilon > 0$ there exists a constant c such that $\chi'_{\mathrm{st}}(K_n) \leq cn^{1+\epsilon}$ for every $n \geq 1$.

Complete graphs

Theorem 25 (Dvořák, Mohar, Šámal, 2013)

The star chromatic index of the complete graph K_n satisfies

$$2n\frac{n-1}{n+2} \le \chi'_{\mathrm{st}}(K_n) \le n\frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{1/4}}.$$

In particular, for every $\epsilon > 0$ there exists a constant c such that $\chi'_{\mathrm{st}}(K_n) \le cn^{1+\epsilon}$ for every $n \ge 1$.

- The lower bound can be improved to $3n\frac{n-1}{n+4}$ using the same argument as Dvořák et al.

The Conjecture

- There is no particularly nice conjecture for general graphs, so the main conjecture in this field is related to complete graphs

The Conjecture

- There is no particularly nice conjecture for general graphs, so the main conjecture in this field is related to complete graphs

Conjecture 26 (Dvořák, Mohar, Šámal, 2013)

The star chromatic index of the complete graph K_n is linear in n, i.e.,

$$\chi'_{\mathrm{st}}(K_n) \in \mathcal{O}(n).$$

Open problems

- Apart from graphs of maximum degree at most 2, we do not know much about this index

Open problems

- Apart from graphs of maximum degree at most 2, we do not know much about this index
- Open for complete graphs, complete bipartite graphs, hypercubes,...

Open problems

- Apart from graphs of maximum degree at most 2, we do not know much about this index
- Open for complete graphs, complete bipartite graphs, hypercubes,...
- The upper bound for general graphs is obtained from the bound for complete graphs

Open problems

- Apart from graphs of maximum degree at most 2, we do not know much about this index
- Open for complete graphs, complete bipartite graphs, hypercubes,...
- The upper bound for general graphs is obtained from the bound for complete graphs

Theorem 27 (Dvořák, Mohar, Šámal, 2013)

For a graph G it holds

$$\chi'_{\mathrm{st}}(G) \leq \chi'_{\mathrm{st}}(K_{\Delta(G)+1}) \cdot O\left(\frac{\log \Delta(G)}{\log\log \Delta(G)}\right)^2,$$

and therefore $\chi'_{\mathrm{st}}(G) \leq \Delta(G) \cdot 2^{O(1)\sqrt{\log \Delta(G)}}$.

Trees and outerplanar graphs

Theorem 28 (Bezegová et al., 2016)

For a tree T it holds

$$\chi'_{\mathrm{st}}(T) \leq \left\lfloor \frac{3\Delta(T)}{2} \right\rfloor,$$

Theorem 29 (Bezegová et al., 2016)

For an outerplanar graph G it holds

$$\chi'_{\mathrm{st}}(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor + 12,$$

Subcubic graphs

- The most analyzed class are subcubic graphs

Subcubic graphs

- The most analyzed class are subcubic graphs

Theorem 30 (Dvořák, Mohar, Šámal, 2013)

(a) *If G is a subcubic graph, then $\chi'_{\mathrm{st}}(G) \leq 7$.*

(b) *If G is a simple cubic graph, then $\chi'_{\mathrm{st}}(G) \geq 4$, and the equality holds if and only if G covers the graph of the 3-cube.*

Subcubic graphs

- The most analyzed class are subcubic graphs

Theorem 30 (Dvořák, Mohar, Šámal, 2013)

(a) *If G is a subcubic graph, then $\chi'_{\mathrm{st}}(G) \leq 7$.*

(b) *If G is a simple cubic graph, then $\chi'_{\mathrm{st}}(G) \geq 4$, and the equality holds if and only if G covers the graph of the 3-cube.*

- There is no known subcubic graph with $\chi'_{\mathrm{st}}(G) = 7$, so Dvořák et al. conjectured that 6 colors suffices

Subcubic graphs

- The most analyzed class are subcubic graphs

Theorem 30 (Dvořák, Mohar, Šámal, 2013)

(a) *If G is a subcubic graph, then $\chi'_{\mathrm{st}}(G) \leq 7$.*

(b) *If G is a simple cubic graph, then $\chi'_{\mathrm{st}}(G) \geq 4$, and the equality holds if and only if G covers the graph of the 3-cube.*

- There is no known subcubic graph with $\chi'_{\mathrm{st}}(G) = 7$, so Dvořák et al. conjectured that 6 colors suffices
- Confirmed for subcubic outerplanar graphs (5 colors)

List version

Question 31 (Dvořák, Mohar, Šámal, 2013)

Is it true that $\mathrm{ch}'_{\mathrm{st}} \leq 7$ *for every subcubic graph* G*? (Perhaps even* ≤ 6*?)*

List version

Question 31 (Dvořák, Mohar, Šámal, 2013)

Is it true that $\mathrm{ch}'_{\mathrm{st}} \leq 7$ *for every subcubic graph* G*? (Perhaps even* ≤ 6*?)*

Theorem 32 (Lužar, Mockovčiaková, Soták, 2017+)

For a subcubic graph G *it holds*

$$\mathrm{ch}'_{\mathrm{st}}(G) \leq 7\,.$$

Dziękuję!