On Incidence Colorings of Graphs

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joint work with

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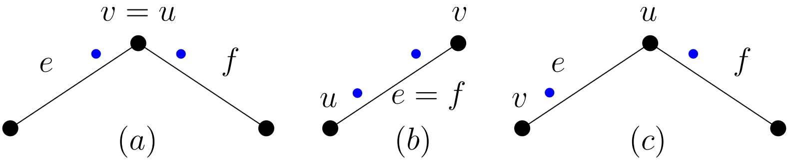
Incidences

- In a graph G, an incidence is a pair (v, e), where $v \in V(G)$, $e \in E(G)$, and v is incident to e.
- Two incidences (v, e) and (u, f) are adjacent if:
  - (a) $v = u$, or
  - (b) $e = f$, or
  - (c) $vu \in \{e, f\}$.



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- Related to other colorings, e.g. strong edge-colorings of fully subdivided graphs.
- Example: C_3

Spectrum

- For an incidence coloring c, the spectrum of a vertex v, $S_c(v)$, is the set of colors assigned to the incidences with the edges containing v, i.e.

$$S_c(v) = \{c(v, uv), c(u, uv) \mid uv \in E(G)\}.$$

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- Spectrum gives us a simple lower bound:

$$\chi_i(G) \geq \min_{c} \max_{v \in V(G)} |S_c(v)| \geq \Delta(G) + 1\,.$$

(k, p)-incidence coloring

- Two types of colors in the spectrum of v:
  - colors of incidences of type (v, uv) - $S_0(v)$;
  - colors of incidences of type (u, uv) - $S_1(v)$.

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- $\chi_i(G) \leq \chi_{i,p}(G)$

$(\Delta + 1)$-graphs

- A $(\Delta + 1)$-graph is every graph G with

$$\chi_i(G) = \chi_{i,1}(G) = \Delta(G) + 1 .$$

- Complete graphs, trees, outerplanar graphs with $\Delta \geq 7$, planar graphs with $\Delta \geq 14$,...
- $\chi_i(G) \leq \chi_{i,1}(G) = \chi(G^2)$

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Theorem 1 (Sun, 2012)

If G is an n-regular graph, then $\chi_i(G) = n + 1$ if and only if $V(G)$ is a disjoint union of $n + 1$ (perfect) dominating sets.

$(\Delta + \ell)$-graphs

- A $(\Delta + \ell)$-graph is every graph G with

$$\chi_i(G) \leq \Delta(G) + \ell\,.$$

- A (k, p)-graph is every graph G with

$$\chi_{i,p}(G) \leq k\,.$$

$(\Delta + 2)$-conjecture

Conjecture 2 (Brualdi & Massey, 1993)

For every graph G

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- Guilduli, 1997 - Conjecture is false
- Paley graphs need $\Delta + \Omega(\log \Delta)$

Theorem 3 (Guilduli, 1997)

For every graph G

$$\chi_i(G) \leq \Delta(G) + 20 \log(\Delta(G)) + 84\,.$$

$(\Delta + 2)$-conjecture

- Conjecture 2 holds for e.g. cubic, partial 2-trees (hence also outerplanar graphs), toroidal grids, planar graphs with girth at least 6 and maximum degree at least 5, complete bipartite graphs,...
- Exists a graph G on 11 vertices with maximum degree 6 and

$$\chi_i(G) = 9\,.$$

- Open for degrees 4 and 5.

Theorem 4

For every graph G with maximum degree 4,

$$\chi_i(G) \leq 7\,.$$

Hypercubes

Theorem 5 (Pai et al., 2014)

For every integers $p, q \geq 1$,

(*i*) $\chi_i(Q_n) = n + 1$, *if* $n = 2^p - 1$;

(*ii*) $\chi_i(Q_n) = n + 2$, *if* $n = 2^p - 2$ *and* $p \geq 2$, *or* $n = 2^p + 2^q - 1$, *or* $n = 2^p + 2^q - 3$ *and* $p, q \geq 2$.

Hypercubes conjecture

Our motivation:

Conjecture 6 (Pai et al., 2014)

For every $n \geq 1$ *such that* $n \neq 2^p - 1$ *for every integer* $p \geq 1$,

$$\chi_i(Q_n) = n + 2\,.$$

Cartesian products

Observation. Let G and H be arbitrary graphs. Then

$$\chi_i(G \,\square\, H) \leq \chi_i(G) + \chi_i(H)\,.$$

- Is it possible that Conjecture 2 holds for Cartesian products?

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- No.

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- Is it possible that Conjecture 2 holds for Cartesian products?
- No.
- Consider a Paley graph P and K_2,

$$\chi_i(P \,\square\, K_2) = \Delta(P \,\square\, K_2) + \Omega(\log(P \,\square\, K_2))\,.$$

Cartesian products

-1 color

Cartesian products (-1 color)

Theorem 7

Let G be a $(\Delta+1)$-graph and let H be a subgraph of a regular $(\Delta+1)$-graph H' such that

$$\Delta(G) + 1 \geq \Delta(H') - \Delta(H).$$

Then,

$$\chi_i(G \,\square\, H) \leq \Delta(G \,\square\, H) + 2\,.$$

Proof of Theorem 7

Proof (1/3).

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- $C = A \cap B = \{t, \ldots, \Delta(G)\}$ - overlapping colors between c and d

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- $C = A \cap B = \{t, \ldots, \Delta(G)\}$ - overlapping colors between c and d
- $|C| = \Delta(G) - t + 1 = \Delta(H') - \Delta(H)$
- $H \subseteq H'$ and $d = d'\big|_H \Rightarrow |S_d(v)| = d(v) + 1 \leq \Delta(H) + 1$

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- $M(v) = B \setminus S_d(v)$ - colors from B missing at v

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- $\Rightarrow$ exists i.c. f of $G \,\square\, H$ with

$$\begin{aligned} |A| + |B| - |C| &= \Delta(G) + 1 + \Delta(H') + 1 - (\Delta(H') - \Delta(H)) \\ &= \Delta(G) + \Delta(H) + 2 \\ &= \Delta(G \,\square\, H) + 2 \end{aligned}$$

colors

Proof of Theorem 7

Proof (3/3).

- For every pair of vertices $u \in V(G)$, $v \in V(H)$, and edges $uu' \in E(G)$, $vv' \in E(H)$, we define f:

$$f\big((u,v),(u,v)(u,v')\big) = d(v,vv'), \quad \text{and}$$

$$f\big((u,v),(u,v)(u',v)\big) = \begin{cases} c(u,uu') & \text{if } c(u,uu') \notin C, \\ g_v(c(u,uu')) & \text{if } c(u,uu') \in C. \end{cases}$$

- We check that f is indeed an i.c. on the board

Hypercubes - revisited

Corollary 8

For every $n \geq 1$,

$$\chi_i(Q_n) = \begin{cases} n+1 & \text{if } n = 2^m - 1 \text{ for some integer } m \geq 0, \\ n+2 & \text{otherwise.} \end{cases}$$

Sketch of the proof: $Q_n = Q_{n-k} \,\square\, Q_k$, with $k \leq n$, and take $n = 2^m - 1 + k$.

Generalization

(Simplified version)

Theorem 9 (Shiau, Shiau, Wang, 2015)

Let G and H be arbitrary graphs and $s(H)$ the maximum size of a spectrum in H. If $\chi_i(G) \geq \chi_i(H) - s(H)$, then

$$\chi_i(G \,\square\, H) \leq \chi_i(G) + s(H)\,.$$

Cartesian products
-2 colors

Locally injective homomorphisms

- A homomorphism f of G to H is a mapping

$$f : V(G) \to V(H)$$

  such that if $uv \in E(G)$, then $f(u)f(v) \in E(H)$.
- A homomorphism f is locally injective if $f(u) \neq f(v)$ for every $v \in V(G)$ and every pair $vu, vw \in E(G)$.
- f is injective on $N(v)$, for every $v \in V(G)$
- locally injective homomorphisms preserve adjacencies of incidences

Locally injective homomorphisms

Theorem 10 (Duffy, 2015)

Let G and H be simple graphs such that G admits a locally injective homomorphism to H. Then

$$\chi_i(G) \leq \chi_i(H).$$

Proposition 11

A graph G admits a $(k,1)$-incidence coloring iff it admits a locally injective homomorphism to K_k.

2-permutable graphs

- K_{2n}^{-} is a complete graph on $2n$ vertices without a perfect matching
- A connected $2d$-regular graph G is 2-permutable if it admits a locally injective homomorphism to K_{2d+2}^{-}.

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- So:
  - G is $(2d+2)$-partite (with partition sets $P_1, \ldots, P_{2d+2}$);
  - For every i, $1 \leq i \leq 2d+2$, exists $\bar{i}$ such that there are no edges between P_i and $P_{\bar{i}}$;
  - Every $v \in P_i$ has at most one neighbor in P_j, $j \notin \{i, \bar{i}\}$.
  - Every 2-permutable graph is a $(\Delta+2, 1)$-graph.

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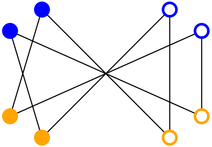
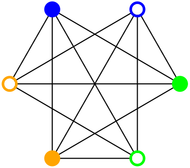
  - Every 2-permutable graph is a $(\Delta + 2, 1)$-graph.
- There exist $(\Delta + 2, 1)$-graphs which are not 2-permutable, e.g. 7-cycle.

2-permutable graphs

Examples: C_{4n} and K_{2n}^{-}





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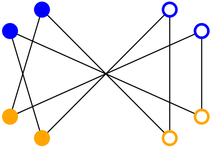

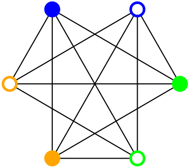

- Among 1544 4-regular graphs of order 12, there are 13 2-permutable graphs.

Prisms over 2-permutable graphs

Theorem 12

Let G be a 2-permutable graph. Then

$$\chi_i(G \,\square\, K_2) = \Delta(G \,\square\, K_2) + 1 \; (= \Delta(G) + 2)\,.$$

Prisms over 2-permutable graphs

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- The inverse of Theorem 13 does not hold in general.

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- The inverse of Theorem 13 does not hold in general.
- The prism over the Dodecahedron is a $(\Delta + 1)$-graph, while the Dodecahedron is not 2-permutable (it is cubic).
- The inverse holds for cycles.

Sub-2-permutable graphs

- A (non-regular) graph G is sub-2-permutable if it admits a locally injective homomorphism to $K^-_{\Delta(G)+2}$.

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Corollary 13

Let G be a sub-2-permutable graph. Then

$$\chi_i(G \,\square\, K_2) = \Delta(G \,\square\, K_2) + 1\,.$$

2-adjustable graphs

- An incidence coloring of a graph G is adjustable if there exists a pair of colors x and y such that there is no vertex $v \in V(G)$ with $x, y \in S_0(v)$.
- x and y are called free colors.

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- Example: C_5

2-adjustable graphs - Examples

- All $(\Delta + 1)$-graphs (the color $\Delta(G) + 2$ is never used).

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2-adjustable graphs - Examples

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- All $(\Delta + 1)$-graphs together with a matching (two same colors can be put on a matching; they are free)
- Cycles, complete bipartite graphs, prisms over C_{6n}

2-adjustable graphs - Examples

- By $\mathring{K}_n$ we denote the complete graph of order n with a loop at one vertex.

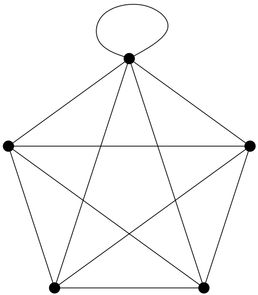

Figure: A $\mathring{K}_5$.

2-adjustable graphs - Examples

Proposition 14

If a graph G admits a locally injective homomorphism to $\mathring{K}_{\Delta(G)+1}$, then G is 2-adjustable.

2-adjustable graphs - Examples

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- The inverse statement is not true in general.
- C_5 is 2-adjustable, but does not admit a locally injective homomorphism to $\mathring{K}_3$.

Cartesian products with -2 colors

Theorem 15

Let G be a sub-2-permutable graph and let H be a 2-adjustable graph. Then

$$\chi_i(G \,\Box\, H) \leq \Delta(G \,\Box\, H) + 2\,.$$

Open problems

Conjecture 16

Let G be a $(\Delta+1)$-graph and H be a $(\Delta+2)$-graph. Then,

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Question 17

Do there exist graphs G and H with $\chi_i(G) = \Delta(G) + 2$ and $\chi_i(H) = \Delta(H) + 2$ such that $\chi_i(G \,\square\, H) > \Delta(G \,\square\, H) + 2$?

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Question 18

When is the Cartesian product of two $(\Delta + 1)$-graphs also a $(\Delta + 1)$-graph.

Hvala!