# On Incidence Colorings of Graphs


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joint work with

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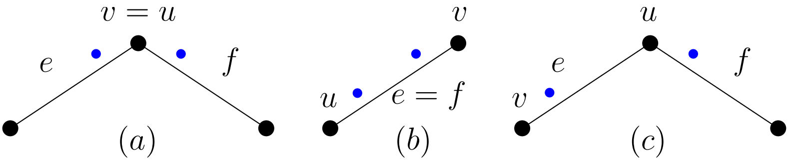
September 5, 2016

# Incidences

- In a graph $G$, an incidence is a pair $(v, e)$, where $v \in V(G)$, $e \in E(G)$, and $v$ is incident to $e$.
- Two incidences $(v, e)$ and $(u, f)$ are adjacent if:
  - $(a)$ $v = u$, or
  - $(b)$ $e = f$, or
  - $(c)$ $vu \in \{e, f\}$.



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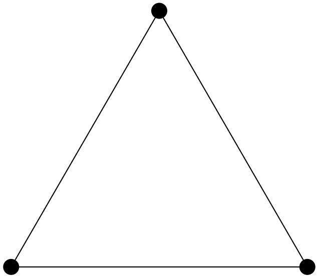
- Defined by Brualdi and Massey in 1993.
- Related to other colorings, e.g. strong edge-coloring of fully subdivided graphs.

# Example: $C_3$



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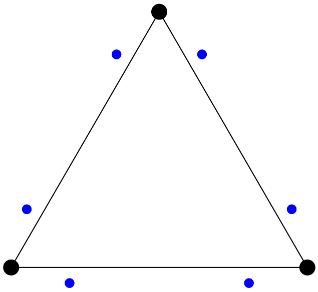

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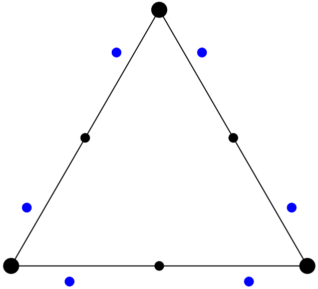

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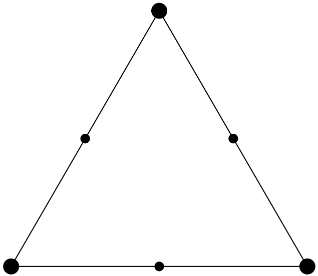

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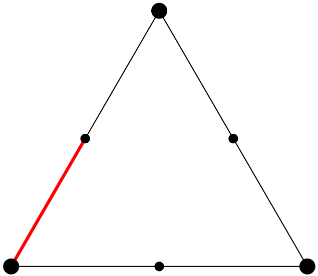

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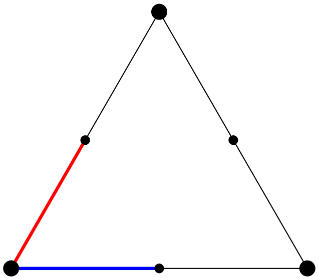

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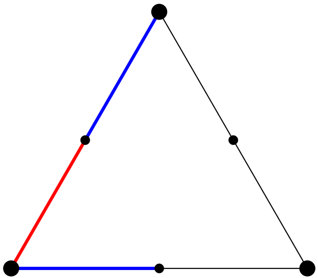

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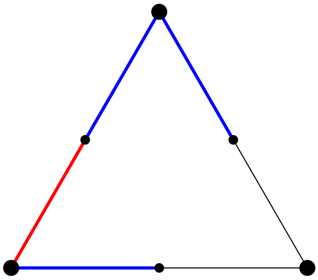

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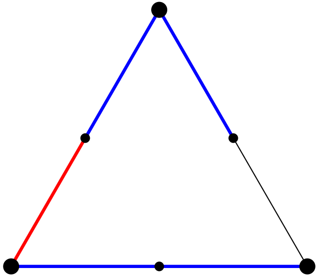

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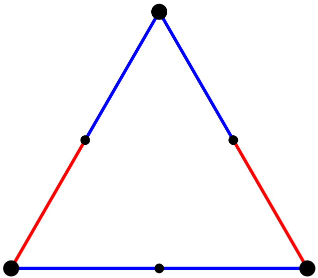

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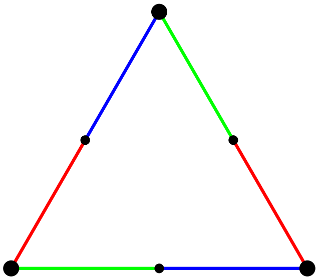

# Spectrum

- For an incidence coloring $c$, the spectrum of a vertex $v$, $S_c(v)$, is the set of colors assigned to the incidences with the edges containing $v$, i.e.

$$S_c(v) = \{c(v, uv), c(u, uv) \mid uv \in E(G)\}.$$

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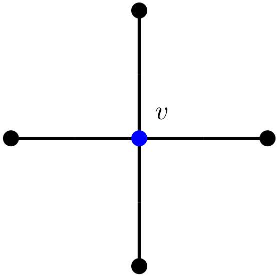

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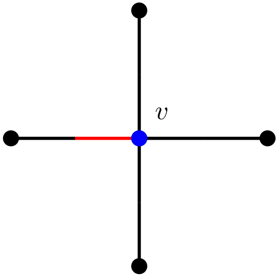

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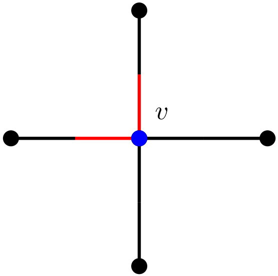

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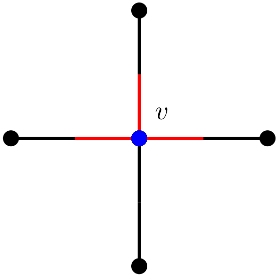

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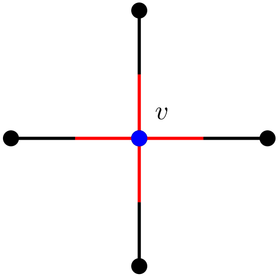

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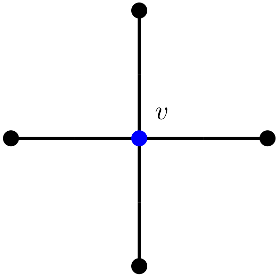

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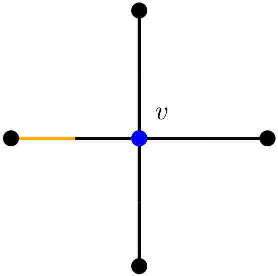

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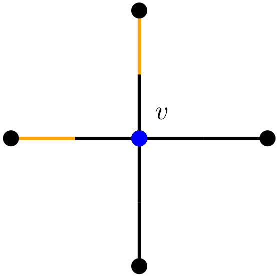

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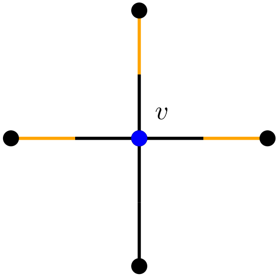

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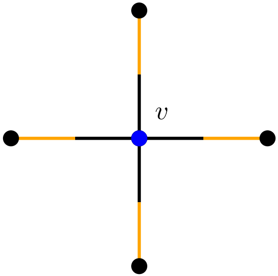

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# Spectrum

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- $S_c^1(v) = \{c(u, uv) \mid uv \in E(G)\}$
- $|S_c^0(v)| = d(v)$ and $|S_c^1(v)| \geq 1$
- Spectrum gives a simple lower bound:

$$\chi_i(G) \geq \min_c \max_{v \in V(G)} |S_c(v)| \geq \Delta(G) + 1\,.$$

# $(\Delta + 1)$-graphs

- A $(\Delta + 1)$-graph is every graph $G$ with

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- Complete graphs, trees, outerplanar graphs with $\Delta \geq 7$, planar graphs with $\Delta \geq 14$,...

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**Theorem 1 (Sun, 2012)**

*If $G$ is an $n$-regular graph, then $\chi_i(G) = n + 1$ if and only if $V(G)$ is a disjoint union of $n + 1$ (perfect) dominating sets.*

# $(\Delta + 2)$-conjecture

**Conjecture 2 (Brualdi & Massey, 1993)**

*For every graph $G$*

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**Theorem 3 (Guilduli, 1997)**

*For every graph $G$*

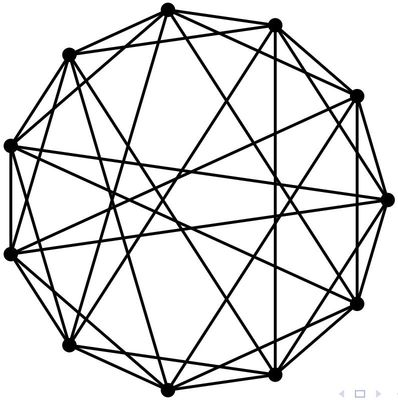
$$\chi_i(G) \leq \Delta(G) + 20\log(\Delta(G)) + 84\,.$$

# $(\Delta + 2)$-conjecture

- Conjecture 2 holds for e.g.
  - subcubic graphs,
  - partial 2-trees (hence also outerplanar graphs),
  - toroidal grids,
  - planar graphs with girth at least 6 and maximum degree at least 5,
  - complete bipartite graphs,
  - ...

# $(\Delta + 2)$-conjecture

The graph $G$ of smallest order being a counter example (Clark & Dunning, 1997):
6-regular, 11 vertices, $\gamma(G) = 3$, $\chi_i(G) = 9$



# $(\Delta + 2)$-conjecture

**Theorem 4 (Maydanskiy, 2005)**

$$\chi_i(G) \geq \frac{2|E(G)|}{|V(G)| - \gamma(G)}.$$

- So far, the only known graphs being counter-examples to the conjecture are the ones having high domination number
- Open for $\Delta(G) \in \{4, 5\}$.
- Strong edge-coloring result gives $\chi_i(G) \leq 2\Delta(G)$ (Nakprasit, 2008)

# Subquartic graphs

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**Question 6**

*Is it true that 6 (resp. 7) colors suffice for incidence coloring of graphs with maximum degree 4 (resp. 5)?*

# Hypercubes

**Theorem 7 (Pai et al., 2014)**

*For every integers $p, q \geq 1$,*

(i) $\chi_i(Q_n) = n + 1$, *if* $n = 2^p - 1$;

(ii) $\chi_i(Q_n) = n + 2$, *if* $n = 2^p - 2$ *and* $p \geq 2$, *or* $n = 2^p + 2^q - 1$, *or* $n = 2^p + 2^q - 3$ *and* $p, q \geq 2$.

# Hypercubes

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Our motivation:

**Conjecture 8 (Pai et al., 2014)**

*For every $n \geq 1$ such that $n \neq 2^p - 1$ for every integer $p \geq 1$,*

$$\chi_i(Q_n) = n + 2\,.$$

# Cartesian products

**Observation.** Let $G$ and $H$ be arbitrary graphs. Then

$$\chi_i(G \,\square\, H) \le \chi_i(G) + \chi_i(H)\,.$$

- Is it possible that Conjecture 2 holds for Cartesian products?

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- Is it possible that Conjecture 2 holds for Cartesian products?
- No.
- Consider a Paley graph $P$ and $K_2$,

$$\chi_i(P \square K_2) = \Delta(P \square K_2) + \Omega(\log(P \square K_2))\,.$$

# Cartesian products
# -1 color

# Cartesian products (−1 color)

**Theorem 9 (Gregor, ·, Soták, 2016)**

*Let $G$ be a $(\Delta+1)$-graph and let $H$ be a subgraph of a regular $(\Delta+1)$-graph $H'$ such that*

$$\Delta(G) + 1 \geq \Delta(H') - \Delta(H).$$

*Then,*

$$\chi_i(G \,\square\, H) \leq \Delta(G \,\square\, H) + 2\,.$$

# Hypercubes - revisited

## Corollary 10 (Gregor, ·, Soták, 2016)

*For every* $n \geq 1$*,*

$$\chi_i(Q_n) = \begin{cases} n+1 & \text{if } n = 2^m - 1 \text{ for some integer } m \geq 0, \\ n+2 & \text{otherwise.} \end{cases}$$

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The conjecture has also been solved independently by Shiau, Shiau, Wang, 2015

# Cartesian products
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Not today :(

# Open problems

## Conjecture 11

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## Question 12

*Do there exist graphs $G$ and $H$ with $\chi_i(G) = \Delta(G) + 2$ and $\chi_i(H) = \Delta(H) + 2$ such that $\chi_i(G \,\Box\, H) > \Delta(G \,\Box\, H) + 2$?*

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## Question 13

*When is the Cartesian product of two $(\Delta+1)$-graphs also a $(\Delta+1)$-graph.*

Thank you for your attention!

# Locally injective homomorphisms

- A homomorphism $f$ of $G$ to $H$ is a mapping

$$f : V(G) \to V(H)$$

  such that if $uv \in E(G)$, then $f(u)f(v) \in E(H)$.
- A homomorphism $f$ is locally injective if $f(u) \neq f(v)$ for every $v \in V(G)$ and every pair $vu, vw \in E(G)$.
- $f$ is injective on $N(v)$, for every $v \in V(G)$
- locally injective homomorphisms preserve adjacencies of incidences

# Locally injective homomorphisms

**Theorem 14 (Duffy, 2015)**

*Let $G$ and $H$ be simple graphs such that $G$ admits a locally injective homomorphism to $H$. Then*

$$\chi_i(G) \leq \chi_i(H).$$

**Proposition 15**

*A graph $G$ admits a $(k,1)$-incidence coloring iff it admits a locally injective homomorphism to $K_k$.*

# 2-permutable graphs

- $K_{2n}^{-}$ is a complete graph on $2n$ vertices without a perfect matching
- A connected $2d$-regular graph $G$ is 2-permutable if it admits a locally injective homomorphism to $K_{2d+2}^{-}$.

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- So:
  - $G$ is $(2d+2)$-partite (with partition sets $P_1, \dots, P_{2d+2}$);
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  - Every $v \in P_i$ has at most one neighbor in $P_j$, $j \notin \{i, \bar{i}\}$.
  - Every 2-permutable graph is a $(\Delta+2, 1)$-graph.

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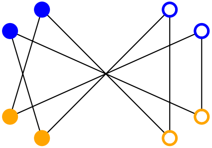
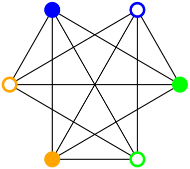
  - Every 2-permutable graph is a $(\Delta+2, 1)$-graph.
- There exist $(\Delta+2, 1)$-graphs which are not 2-permutable, e.g. 7-cycle.

# 2-permutable graphs

Examples: $C_{4n}$ and $K_{2n}^{-}$





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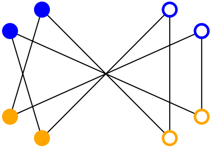

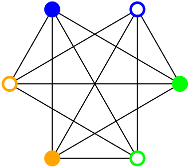

- Among 1544 4-regular graphs of order 12, there are 13 2-permutable graphs.

# Prisms over 2-permutable graphs

**Theorem 16**

*Let $G$ be a 2-permutable graph. Then*

$$\chi_i(G \square K_2) = \Delta(G \square K_2) + 1 \;(= \Delta(G) + 2)\,.$$

# Prisms over 2-permutable graphs

## Theorem 16

*Let $G$ be a 2-permutable graph. Then*

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- The inverse of Theorem 16 does not hold in general.
- The prism over the Dodecahedron is a $(\Delta + 1)$-graph, while the Dodecahedron is not 2-permutable (it is cubic).
- The inverse holds for cycles.

# Sub-2-permutable graphs

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**Corollary 17**

*Let $G$ be a sub-2-permutable graph. Then*

$$\chi_i(G \,\square\, K_2) = \Delta(G \,\square\, K_2) + 1\,.$$

# 2-adjustable graphs

- An incidence coloring of a graph $G$ is adjustable if there exists a pair of colors $x$ and $y$ such that there is no vertex $v \in V(G)$ with $x, y \in S_0(v)$.
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- A graph $G$ is 2-adjustable if it admits an adjustable $(\Delta(G) + 2)$-incidence coloring.
- Example: $C_5$

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- All $(\Delta + 1)$-graphs (the color $\Delta(G) + 2$ is never used).

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- All $(\Delta + 1)$-graphs (the color $\Delta(G) + 2$ is never used).
- All $(\Delta + 1)$-graphs together with a matching (two same colors can be put on a matching; they are free)
- Cycles, complete bipartite graphs, prisms over $C_{6n}$

# 2-adjustable graphs - Examples

- By $\mathring{K}_n$ we denote the complete graph of order $n$ with a loop at one vertex.

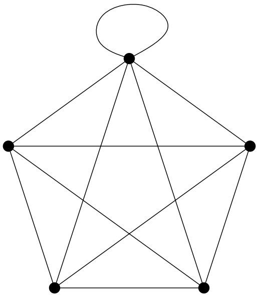

Figure: A $\mathring{K}_5$.

# 2-adjustable graphs - Examples

**Proposition 18**

*If a graph $G$ admits a locally injective homomorphism to $\mathring{K}_{\Delta(G)+1}$, then $G$ is 2-adjustable.*

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- The inverse statement is not true in general.
- $C_5$ is 2-adjustable, but does not admit a locally injective homomorphism to $\mathring{K}_3$.

# Cartesian products with −2 colors

**Theorem 19**

*Let $G$ be a sub-2-permutable graph and let $H$ be a 2-adjustable graph. Then*

$$\chi_i(G \,\square\, H) \leq \Delta(G \,\square\, H) + 2\,.$$

# Open problems

## Conjecture 20

*Let $G$ be a $(\Delta+1)$-graph and $H$ be a $(\Delta+2)$-graph. Then,*

$$\chi_i(G \,\Box\, H) \leq \Delta(G \,\Box\, H) + 2\,.$$

# Open problems

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## Question 21

*Do there exist graphs $G$ and $H$ with $\chi_i(G) = \Delta(G) + 2$ and $\chi_i(H) = \Delta(H) + 2$ such that $\chi_i(G \,\Box\, H) > \Delta(G \,\Box\, H) + 2$?*

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## Question 22

*When is the Cartesian product of two $(\Delta+1)$-graphs also a $(\Delta+1)$-graph.*

Thank you for your attention!