On Incidence Colorings of Graphs

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joint work with

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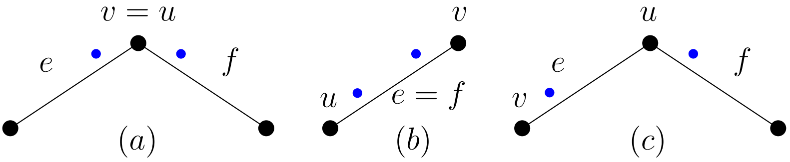
Incidences

- In a graph G, an incidence is a pair (v, e), where $v \in V(G)$, $e \in E(G)$, and v is incident to e.
- Two incidences (v, e) and (u, f) are adjacent if:
  - (a) $v = u$, or
  - (b) $e = f$, or
  - (c) $vu \in \{e, f\}$.



Incidence coloring

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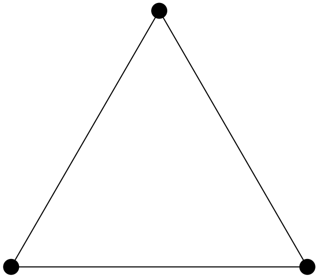
- Defined by Brualdi and Massey in 1993.
- Related to other colorings, e.g. strong edge-coloring of fully subdivided graphs.

Example: C_3



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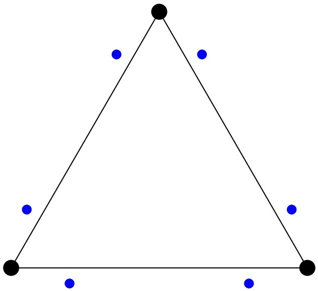

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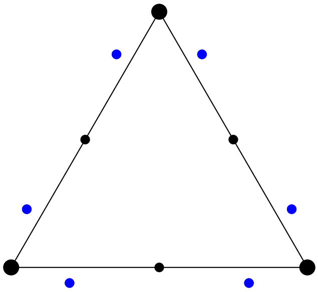

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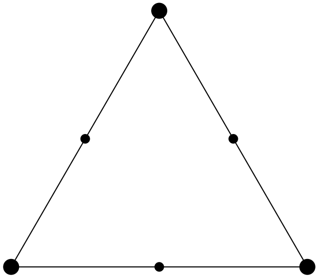

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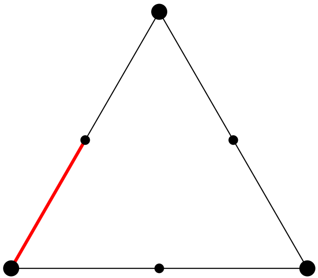

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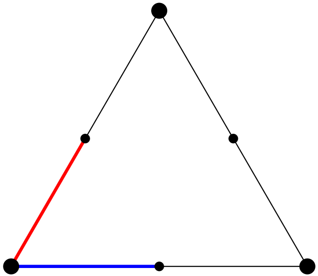

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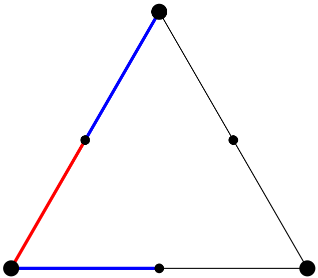

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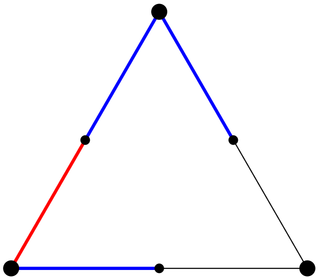

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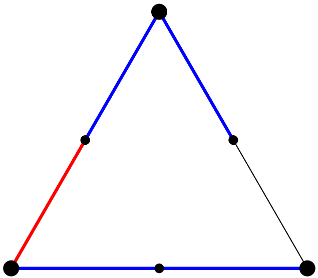

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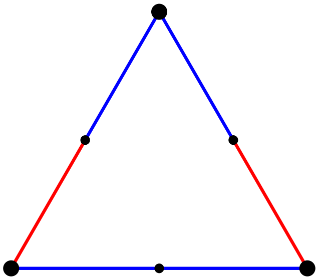

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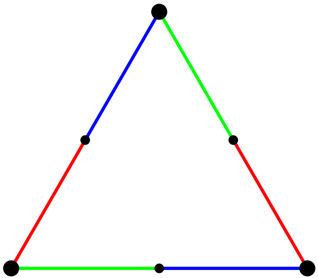

Spectrum

- For an incidence coloring c, the spectrum of a vertex v, $S_c(v)$, is the set of colors assigned to the incidences with the edges containing v, i.e.

$$S_c(v) = \{c(v, uv), c(u, uv) \mid uv \in E(G)\}.$$

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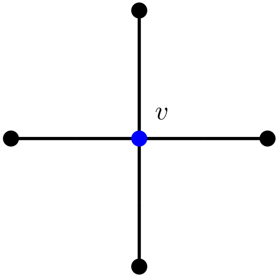

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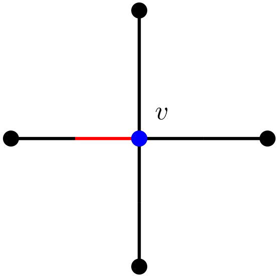

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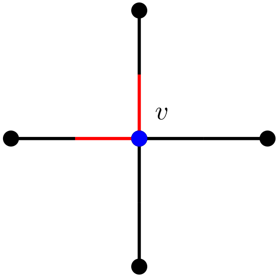

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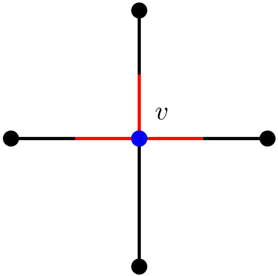

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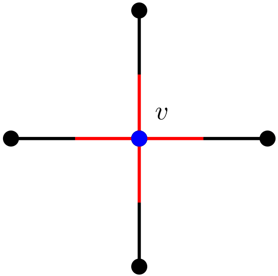

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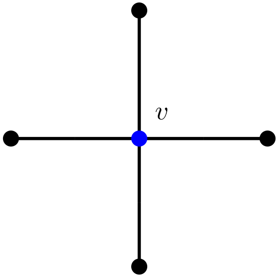

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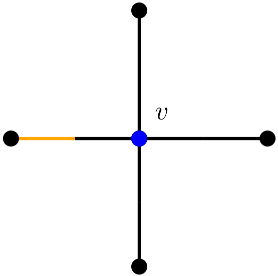

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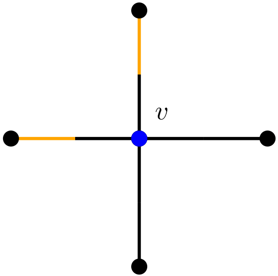

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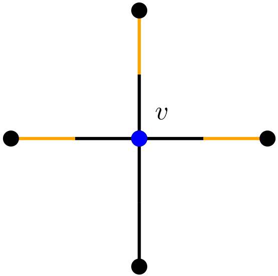

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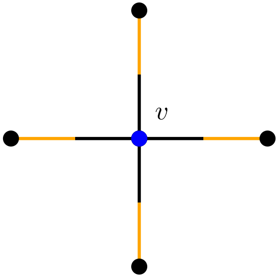

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Spectrum

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- $S_c^1(v) = \{c(u, uv) \mid uv \in E(G)\}$
- $|S_c^0(v)| = d(v)$ and $|S_c^1(v)| \geq 1$
- Spectrum gives a simple lower bound:

$$\chi_i(G) \geq \min_c \max_{v \in V(G)} |S_c(v)| \geq \Delta(G) + 1\,.$$

(k, p)-incidence coloring

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- $\chi_i(G) \leq \chi_{i,p}(G)$

$(\Delta + 1)$-graphs

- A $(\Delta + 1)$-graph is every graph G with

$$\chi_i(G) = \Delta(G) + 1\,.$$

- Complete graphs, trees, outerplanar graphs with $\Delta \geq 7$, planar graphs with $\Delta \geq 14$,...

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Theorem 1 (Sun, 2012)

If G is an n-regular graph, then $\chi_i(G) = n + 1$ if and only if $V(G)$ is a disjoint union of $n + 1$ (perfect) dominating sets.

$(\Delta + \ell)$-graphs

- A $(\Delta + \ell)$-graph is every graph G with

$$\chi_i(G) \leq \Delta(G) + \ell\,.$$

- A (k, p)-graph is every graph G with

$$\chi_{i,p}(G) \leq k\,.$$

$(\Delta + 2)$-conjecture

Conjecture 2 (Brualdi & Massey, 1993)

For every graph G

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Theorem 3 (Guilduli, 1997)

For every graph G

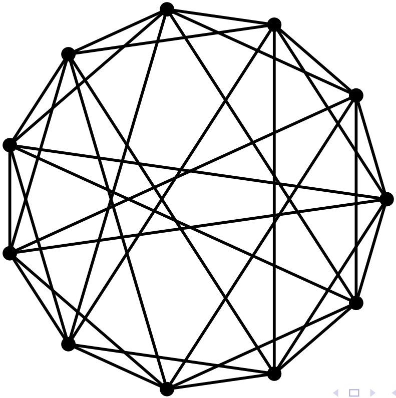
$$\chi_i(G) \leq \Delta(G) + 20\log(\Delta(G)) + 84\,.$$

$(\Delta + 2)$-conjecture

- Conjecture 2 holds for e.g.
  - subcubic graphs,
  - partial 2-trees (hence also outerplanar graphs),
  - toroidal grids,
  - planar graphs with girth at least 6 and maximum degree at least 5,
  - complete bipartite graphs,
  - ...

$(\Delta + 2)$-conjecture

The graph G of smallest order being a counter example (Clark & Dunning, 1997):
6-regular, 11 vertices, $\gamma(G) = 3$, $\chi_i(G) = 9$



$(\Delta + 2)$-conjecture

Theorem 4 (Maydanskiy, 2005)

$$\chi_i(G) \geq \frac{2|E(G)|}{|V(G)| - \gamma(G)}.$$

- So far, the only known graphs being counter-examples to the conjecture are the ones having high domination number
- Open for $\Delta(G) \in \{4, 5\}$.
- Strong edge-coloring result gives $\chi_i(G) \leq 2\Delta(G)$ (Brualdi & Massey, 1993)

Subquartic graphs

Theorem 5 (Gregor, BL, Soták, 2016)

For every graph G with maximum degree 4,

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- [many 4-regular graphs on 15 vertices are $(\Delta + 2)$-graphs] :)

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- [many 4-regular graphs on 15 vertices are $(\Delta + 2)$-graphs] :)

Question 6

Is it true that 6 (resp. 7) colors suffice for incidence coloring of graphs with maximum degree 4 (resp. 5)?

Hypercubes

Theorem 7 (Pai et al., 2014)

For every integers $p, q \geq 1$,

(i) $\chi_i(Q_n) = n + 1$, *if* $n = 2^p - 1$;

(ii) $\chi_i(Q_n) = n + 2$, *if* $n = 2^p - 2$ *and* $p \geq 2$, *or* $n = 2^p + 2^q - 1$, *or* $n = 2^p + 2^q - 3$ *and* $p, q \geq 2$.

Hypercubes

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Our motivation:

Conjecture 8 (Pai et al., 2014)

For every $n \geq 1$ such that $n \neq 2^p - 1$ for every integer $p \geq 1$,

$$\chi_i(Q_n) = n + 2 \, .$$

Cartesian products

Observation. Let G and H be arbitrary graphs. Then

$$\chi_i(G \,\square\, H) \leq \chi_i(G) + \chi_i(H)\,.$$

- Is it possible that Conjecture 2 holds for Cartesian products?

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- Is it possible that Conjecture 2 holds for Cartesian products?
- No.
- Consider a Paley graph P and K_2,

$$\chi_i(P \,\square\, K_2) = \Delta(P \,\square\, K_2) + \Omega(\log(P \,\square\, K_2))\,.$$

Cartesian products
-1 color

$$\chi_i(G \,\square\, H) \leq \chi_i(G) + \chi_i(H) - 1$$

Cartesian products (−1 color)

Theorem 9 (Gregor, BL, Soták, 2016)

Let G be a $(\Delta+1)$-graph and let H be a subgraph of a regular $(\Delta+1)$-graph H' such that

$$\Delta(G) + 1 \geq \Delta(H') - \Delta(H).$$

Then,

$$\chi_i(G \,\square\, H) \leq \Delta(G \,\square\, H) + 2\,.$$

Hypercubes - revisited

Corollary 10 (Gregor, BL, Soták, 2016)

For every $n \geq 1$,

$$\chi_i(Q_n) = \begin{cases} n+1 & \text{if } n = 2^m - 1 \text{ for some integer } m \geq 0, \\ n+2 & \text{otherwise.} \end{cases}$$

Hypercubes - revisited

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The conjecture has also been solved independently by Shiau, Shiau, Wang, 2015

Cartesian products
-2 colors

$$\chi_i(G \,\square\, H) \leq \chi_i(G) + \chi_i(H) - 2$$

Locally injective homomorphisms

- A homomorphism f of G to H is a mapping

$$f : V(G) \to V(H)$$

  such that if $uv \in E(G)$, then $f(u)f(v) \in E(H)$.
- A homomorphism f is locally injective if $f(u) \neq f(v)$ for every $v \in V(G)$ and every pair $vu, vw \in E(G)$.
- f is injective on $N(v)$, for every $v \in V(G)$
- locally injective homomorphisms preserve adjacencies of incidences

Locally injective homomorphisms

Theorem 11 (Duffy, 2015)

Let G and H be simple graphs such that G admits a locally injective homomorphism to H. Then

$$\chi_i(G) \leq \chi_i(H) .$$

Proposition 12

A graph G admits a $(k, 1)$-incidence coloring iff it admits a locally injective homomorphism to K_k.

2-permutable graphs

- K_{2n}^{-} is a complete graph on $2n$ vertices without a perfect matching
- A connected $2d$-regular graph G is 2-permutable if it admits a locally injective homomorphism to K_{2d+2}^{-}.

2-permutable graphs

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- A connected $2d$-regular graph G is 2-permutable if it admits a locally injective homomorphism to K_{2d+2}^-.
- So:
  - G is $(2d+2)$-partite (with partition sets $P_1, \ldots, P_{2d+2}$);
  - For every i, $1 \leq i \leq 2d+2$, exists $\bar{i}$ such that there are no edges between P_i and $P_{\bar{i}}$;
  - Every $v \in P_i$ has at most one neighbor in P_j, $j \notin \{i, \bar{i}\}$.
  - Every 2-permutable graph is a $(\Delta + 2, 1)$-graph.

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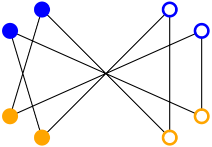
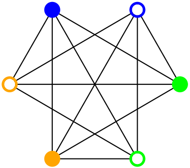
  - Every 2-permutable graph is a $(\Delta + 2, 1)$-graph.
- There exist $(\Delta + 2, 1)$-graphs which are not 2-permutable, e.g. 7-cycle.

2-permutable graphs

Examples: C_{4n} and K_{2n}^{-}





2-permutable graphs

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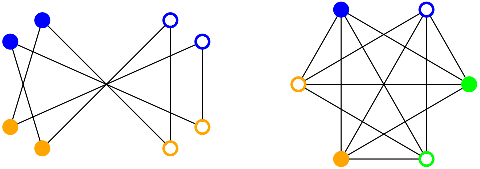

- Among 1544 4-regular graphs of order 12, there are 13 2-permutable graphs.

Prisms over 2-permutable graphs

Theorem 13

Let G be a 2-permutable graph. Then

$$\chi_i(G \,\square\, K_2) = \Delta(G \,\square\, K_2) + 1 \ (= \Delta(G) + 2)\,.$$

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- The prism over the Dodecahedron is a $(\Delta + 1)$-graph, while the Dodecahedron is not 2-permutable (it is cubic).

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- The inverse of Theorem 13 does not hold in general.
- The prism over the Dodecahedron is a $(\Delta + 1)$-graph, while the Dodecahedron is not 2-permutable (it is cubic).
- The inverse holds for cycles.

Sub-2-permutable graphs

- A (non-regular) graph G is sub-2-permutable if it admits a locally injective homomorphism to $K^{-}_{\Delta(G)+2}$.

Sub-2-permutable graphs

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Corollary 14

Let G be a sub-2-permutable graph. Then

$$\chi_i(G \square K_2) = \Delta(G \square K_2) + 1 \,.$$

2-adjustable graphs

- An incidence coloring of a graph G is adjustable if there exists a pair of colors x and y such that there is no vertex $v \in V(G)$ with $x, y \in S_0(v)$.
- x and y are called free colors.

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- Example: C_5

2-adjustable graphs - Examples

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2-adjustable graphs - Examples

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- All $(\Delta + 1)$-graphs together with a matching (two same colors can be put on a matching; they are free)
- Cycles, complete bipartite graphs, prisms over C_{6n}

2-adjustable graphs - Examples

- By $\mathring{K}_n$ we denote the complete graph of order n with a loop at one vertex.

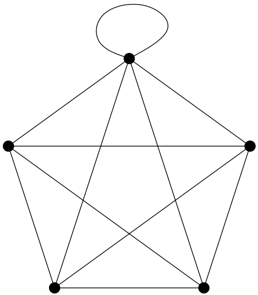

Figure: A $\mathring{K}_5$.

2-adjustable graphs - Examples

Proposition 15

If a graph G admits a locally injective homomorphism to $\mathring{K}_{\Delta(G)+1}$, then G is 2-adjustable.

2-adjustable graphs - Examples

Proposition 15

If a graph G admits a locally injective homomorphism to $\mathring{K}_{\Delta(G)+1}$, then G is 2-adjustable.

- The inverse statement is not true in general.
- C_5 is 2-adjustable, but does not admit a locally injective homomorphism to $\mathring{K}_3$.

Cartesian products with −2 colors

Theorem 16

Let G be a sub-2-permutable graph and let H be a 2-adjustable graph. Then

$$\chi_i(G \,\square\, H) \leq \Delta(G \,\square\, H) + 2\,.$$

Open problems

Conjecture 17

Let G be a $(\Delta+1)$-graph and H be a $(\Delta+2)$-graph. Then,

$$\chi_i(G \,\square\, H) \leq \Delta(G \,\square\, H) + 2\,.$$

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Question 18

Do there exist graphs G and H with $\chi_i(G) = \Delta(G) + 2$ and $\chi_i(H) = \Delta(H) + 2$ such that $\chi_i(G \,\square\, H) > \Delta(G \,\square\, H) + 2$?

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Question 19

When is the Cartesian product of two $(\Delta+1)$-graphs also a $(\Delta+1)$-graph.

Dzięki!