Results on locally irregular edge-coloring

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joint work with

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Basics

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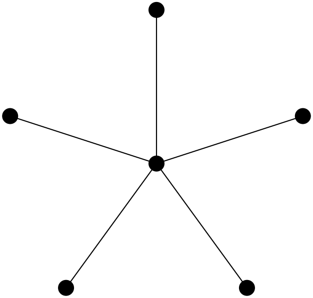
- An edge-coloring is locally irregular if every color class induces a locally irregular graph.
- Always improper—paths of odd length do not admit such a coloring
- Introduced by Baudon, Bensmail, Przybyło, and Woźniak in 2013 (the paper published in 2015).
- Motivated by the (1-2-3)-conjecture:

  For every graph with no K_2 component there exists an edge weighting with 1, 2, and 3 such that for every two adjacent vertices the sums on their incident edges are distinct.

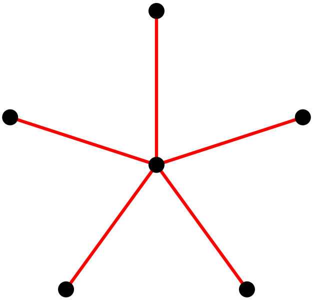
Example: K_5

- A test for the audience... How many colors?



Example: K_5

- Correct!



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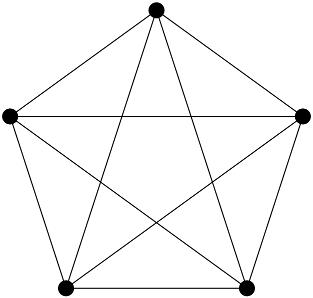

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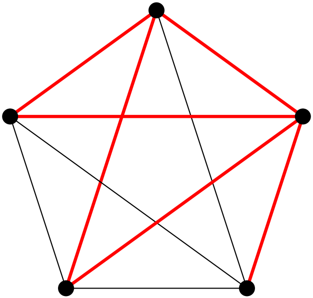

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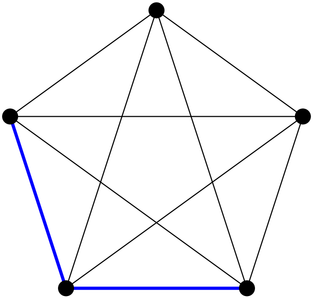

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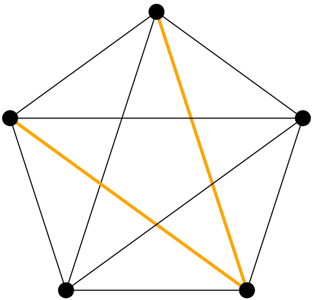

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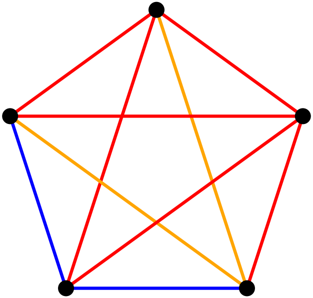

Decomposable graphs

- A graph is decomposable if it admits a locally irregular edge-coloring (LIE-C).
- The minimum k for which there is a LIE-C of a graph G with k colors is the locally irregular chromatic index of G, $\chi'_{\mathrm{irr}}(G)$.

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- The minimum k for which there is a LIE-C of a graph G with k colors is the locally irregular chromatic index of G, $\chi'_{\mathrm{irr}}(G)$.
- Not all graphs are decomposable, e.g. odd-length paths, odd-length cycles.
- A complete characterization was given by Baudon, Bensmail, Przybyło, and Woźniak.

Decomposable graphs

Define a family of graphs $\mathcal{T}$ recursively:

- The triangle C_3 belongs to $\mathcal{T}$.
- Every other graph of this family can be constructed by taking an auxiliary graph F which might either be a path of even length or a path of odd length with a triangle glued to one end, then choosing a graph $G \in \mathcal{T}$ containing a triangle with at least one vertex v of degree 2 and finally identifying v with a vertex of degree 1 in F.

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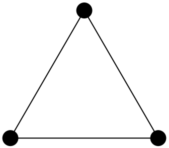

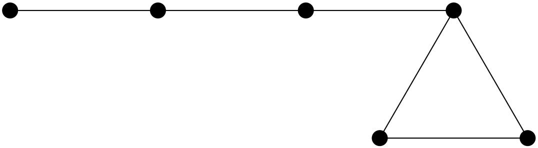

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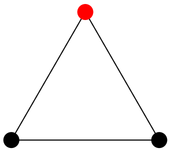

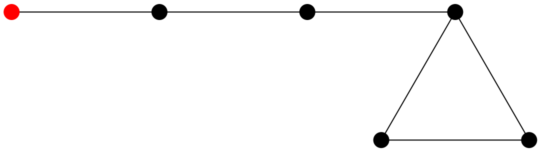

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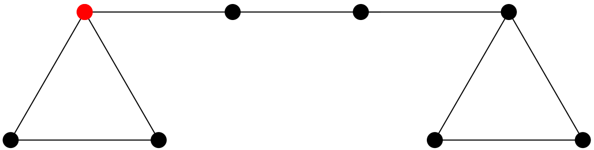

The Conjecture

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Theorem 2 (Baudon et al., 2015)

For every d-regular graph G, with $d \geq 10^7$, it holds $\chi'_{\mathrm{irr}}(G) \leq 3$.

Theorem 3 (Przybyło, 2016+)

For every graph G, with $\delta(G) \geq 10^{10}$, it holds $\chi'_{\mathrm{irr}}(G) \leq 3$.

The upper bound

Bensmail, Merker, and Thomassen established the first constant upper bound using decompositions into bipartite graphs.

Theorem 4 (Bensmail et al., 2017)

For every decomposable graph G, it holds $\chi'_{\mathrm{irr}}(G) \leq 328$.

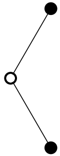
Currently the best:

Theorem 5 (BL, Przybyło, Soták, 2016+)

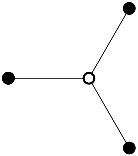
For every decomposable graph G, it holds $\chi'_{\mathrm{irr}}(G) \leq 220$.

Structural lemma

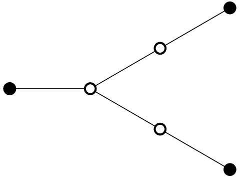
- Let $K''_{1,3}$ denote the complete bipartite graph $K_{1,3}$ with two edges subdivided once.
- An edge-decomposition of a connected graph is pertinent if it is comprised of 2-paths and at most one element isomorphic either to $K_{1,3}$ or $K''_{1,3}$.
- If a graph is not connected, then its edge-decomposition is pertinent if the restriction to every component of the graph is pertinent.



P_3



$K_{1,3}$



$K''_{1,3}$

Structural lemma

Theorem 6 (Bensmail et al., 2017)

Let G be a decomposable connected graph of odd size. Then it contains a locally irregular subgraph H such that every connected component of $G - E(H)$ has even size.

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From the proof of above, we obtain the following formulation:

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Every connected decomposable graph admits a pertinent edge-decomposition.

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Lemma 7 (Bensmail et al., 2017)

Every connected decomposable graph admits a pertinent edge-decomposition.

An edge-decomposition $\mathcal{D}$ of a graph is strongly pertinent if it is pertinent and in the case $\mathcal{D}$ contains an element isomorphic to $K''_{1,3}$ in some component C, the graph has no pertinent edge-decomposition without $K''_{1,3}$ in C.

Subcubic graphs

Theorem 8 (BL, Przybyło, Soták, 2016+)

For every decomposable graph G with $\Delta(G) = 3$, it holds $\chi'_{\mathrm{irr}}(G) \leq 4$.

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Or stronger:

Theorem 9 (BL, Przybyło, Soták, 2016+)

Let G be a decomposable subcubic graph and let $\mathcal{D}$ be a strongly pertinent edge-decomposition of G. Then, G admits a locally irregular edge-coloring with at most 4 colors such that

(i) the edges of every element of $\mathcal{D}$ are colored with the same color; and

(ii) if the edges of two incident elements p_1, p_2 of $\mathcal{D}$ are colored with the same color, then the vertex, at which p_1 and p_2 are incident, is the central vertex of either p_1 or p_2.

Bipartite graphs

Theorem 10 (Baudon et al., 2015)

Let G be a regular bipartite graph with minimum degree at least 3. Then

$$\chi'_{\mathrm{irr}}(G) \leq 2\,.$$

A decomposable bipartite graph is balanced if all the vertices in one of the two partition parts have even degrees.

Lemma 11 (Bensmail et al., 2017)

Let F be a balanced forest. Then F admits a LIE-C with at most 2 colors. Moreover, for each vertex v in the partition with no vertex of odd degree, all edges incident to v have the same color.

Bipartite graphs

Theorem 12 (BL, Przybyło, Soták, 2016+)

Let G be a (multi)graph not isomorphic to an odd cycle. Then

$$\chi'_{\mathrm{irr}}(\mathcal{S}(G)) \leq 2\,.$$

Here, $\mathcal{S}(G)$ denotes the full subdivision of G, i.e. each edge of G is subdivided once.

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Question 13

Is every connected balanced graph, which is not a cycle of length $4k+2$, locally irregular 2-edge-colorable?

Bipartite graphs

Theorem 14 (Bensmail et al., 2017)

Let G be a balanced graph. Then

$$\chi'_{\mathrm{irr}}(G) \leq 7\,.$$

And consequently:

Theorem 15 (Bensmail et al., 2017)

Let G be a decomposable bipartite graph. Then

$$\chi'_{\mathrm{irr}}(G) \leq 10\,.$$

Moreover, if G has an even number of edges, then $\chi'_{\mathrm{irr}}(G) \leq 9$.

Vertex-parity edge-coloring

- $\pi : V(G) \to \{0,1\}$ is a *vertex signature* for G, and a pair (G,π) is a *parity pair*.

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- A *vertex-parity edge-coloring* of a parity pair (G,π) is a (not necessarily proper) edge-coloring such that at every vertex v each appearing color c is in parity accordance with π, i.e. the number of edges of color c incident to v is even if $\pi(v) = 0$, and odd if $\pi(v) = 1$.

Vertex-parity edge-coloring

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- *vertex-parity chromatic index* $\chi'_p(G,\pi)$
- Necessary conditions for the existence of $\chi'_p(G,\pi)$:

  (P_1) Every vertex v of (G,π) with $\pi(v) = 0$ has even degree in G.

  (P_2) In every component of G, there are zero or at least two vertices with the vertex signature value 1.

Bipartite graphs

Theorem 16 (BL, Petruševski, Škrekovski, 2016+)

Let G be a connected graph, and let (G,π) be a proper parity pair. If $|\pi^{-1}(1)| \neq 3$, then

$$\chi'_p(G,\pi) \leq 4\,.$$

Bipartite graphs

Theorem 16 (BL, Petruševski, Škrekovski, 2016+)

Let G be a connected graph, and let (G, π) be a proper parity pair. If $|\pi^{-1}(1)| \neq 3$, then

$$\chi'_p(G, \pi) \leq 4\,.$$

Consequently:

Theorem 17 (BL, Przybyło, Soták, 2016+)

Let G be a balanced graph. Then

$$\chi'_{\mathrm{irr}}(G) \leq 4\,.$$

Bipartite graphs

Theorem 18 (BL, Przybyło, Soták, 2016+)

Let G be a decomposable bipartite graph. Then

$$\chi'_{\mathrm{irr}}(G) \leq 7\,.$$

Moreover, if G has an even number of edges, then the upper bound is 6.

Even-sized graphs

Theorem 19 (Bensmail et al., 2017)

Let G be a d-degenerate graph of even size. Then

$$\chi'_{\mathrm{irr}}(G) \leq 9(\lceil \log_2(d+1) \rceil + 1)\,.$$

Even-sized graphs

Theorem 19 (Bensmail et al., 2017)

Let G be a d-degenerate graph of even size. Then

$$\chi'_{\mathrm{irr}}(G) \leq 9(\lceil \log_2(d+1) \rceil + 1).$$

By Theorem 18 we hence have:

Corollary 20 (BL, Przybyło, Soták, 2016+)

Let G be a d-degenerate graph of even size. Then

$$\chi'_{\mathrm{irr}}(G) \leq 6(\lceil \log_2(d+1) \rceil + 1).$$

Another structural lemma

Lemma 21 (Bensmail et al., 2017)

Let d be a natural number. If G is a connected graph of even size, then G can be decomposed into two graphs D and H such that D is $2d$-degenerate, every component of D has even size, and the minimum degree of H is at least $d - 1$.

Another structural lemma

Lemma 21 (Bensmail et al., 2017)

Let d be a natural number. If G is a connected graph of even size, then G can be decomposed into two graphs D and H such that D is $2d$-degenerate, every component of D has even size, and the minimum degree of H is at least $d-1$.

This together with Theorem 3 implies the main theorem:

Theorem 22 (BL, Przybyło, Soták, 2016+)

For every decomposable graph G, it holds $\chi'_{\mathrm{irr}}(G) \leq 220$.

Sketch of the Proof

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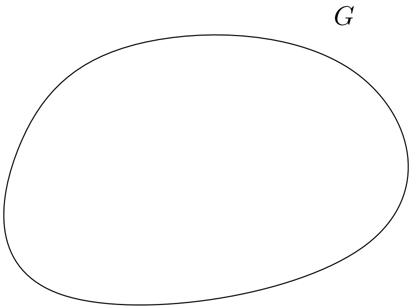

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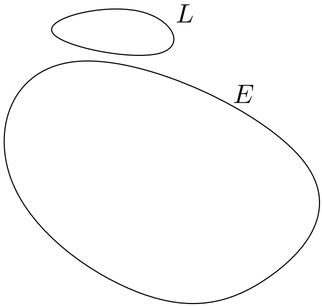

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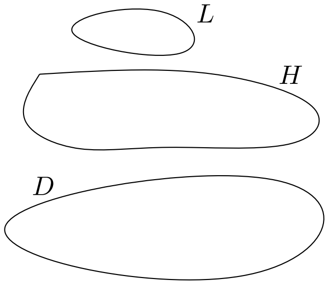

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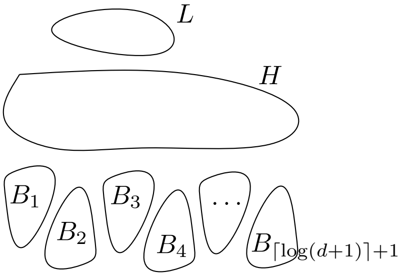

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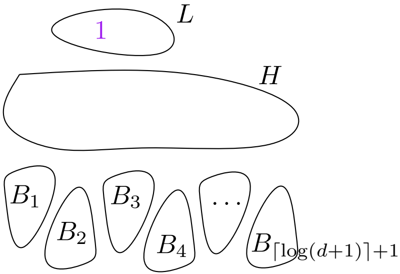

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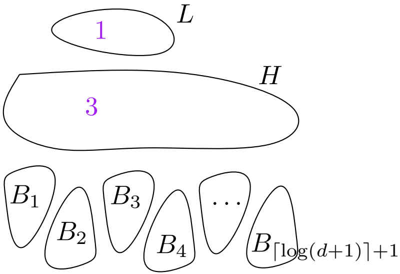

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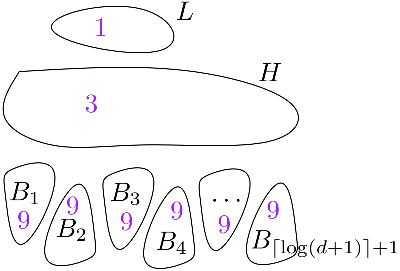

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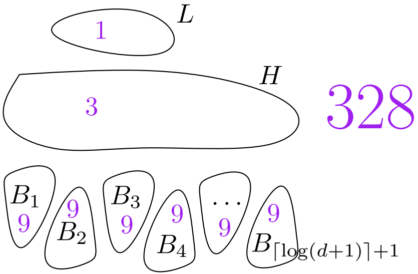

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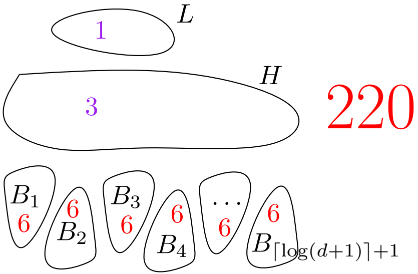

Děkuju!

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Ďakujem!