# On 3-Choosability of Planar Graphs with Maximum Degree 4

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joint work with
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# The Problem

**Problem 1 (Czap, Jendroľ & Voigt [3])**

*Is there a bipartite plane graph such that its medial graph has chromatic number 4?*

In other words:

Is there a bipartite plane graph that needs 4 colors for facially-proper edge-coloring?

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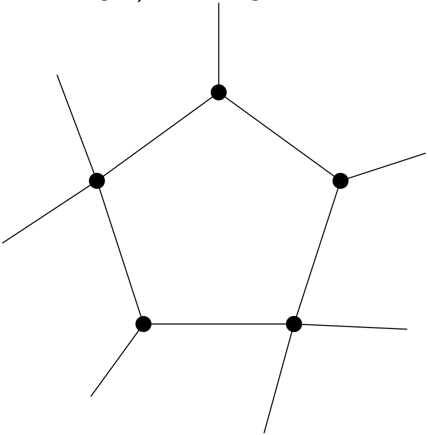

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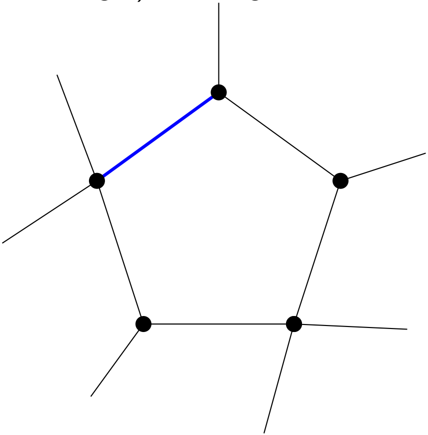

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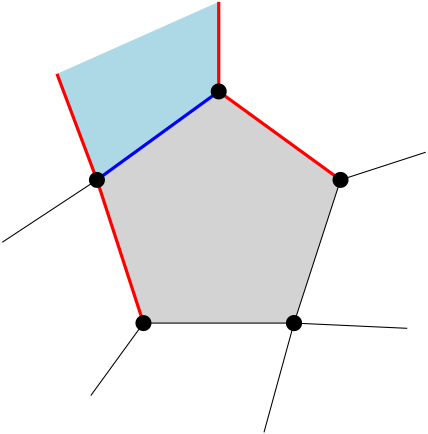

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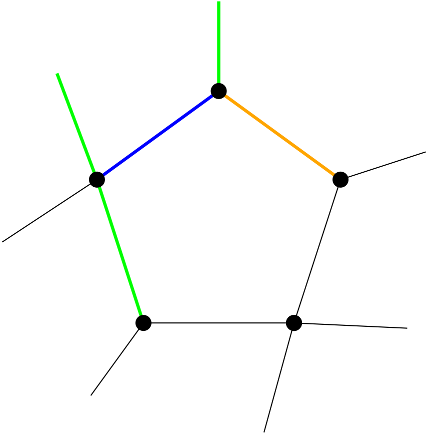

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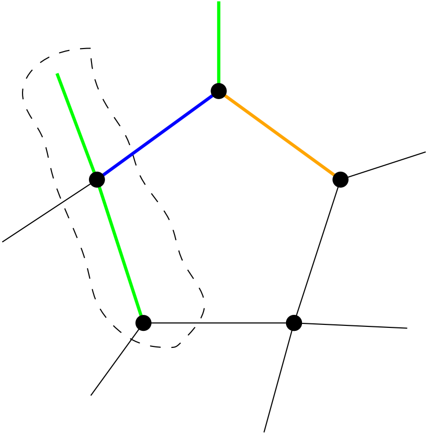

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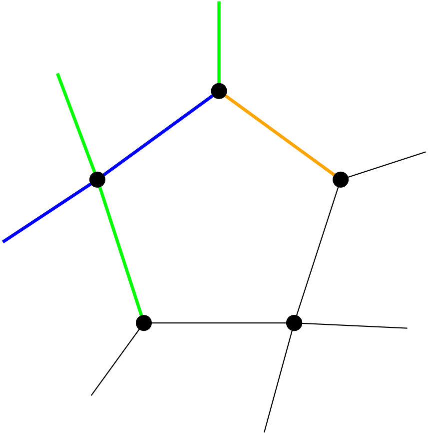

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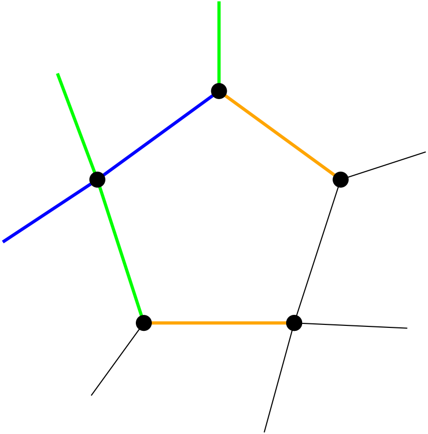

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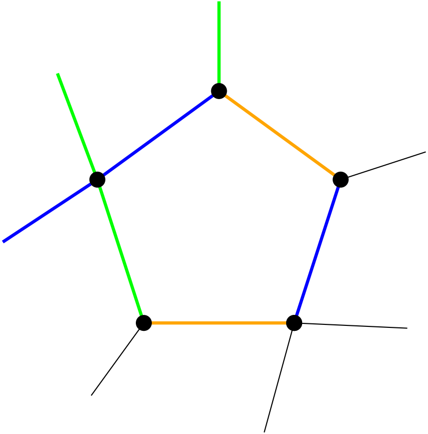

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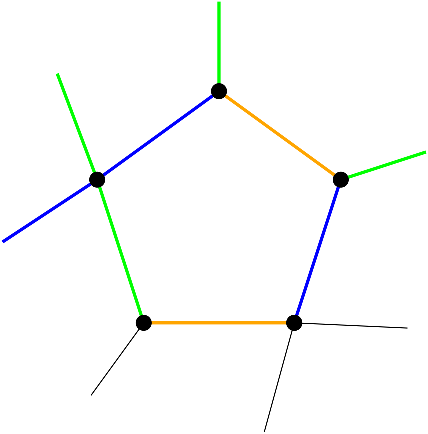

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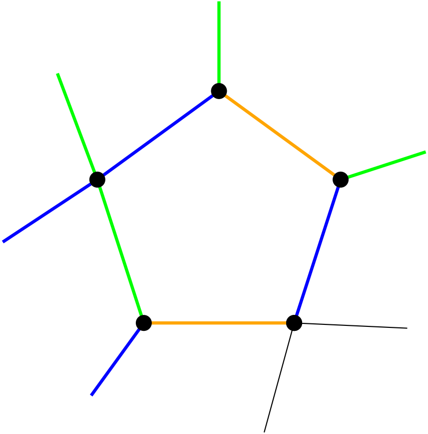

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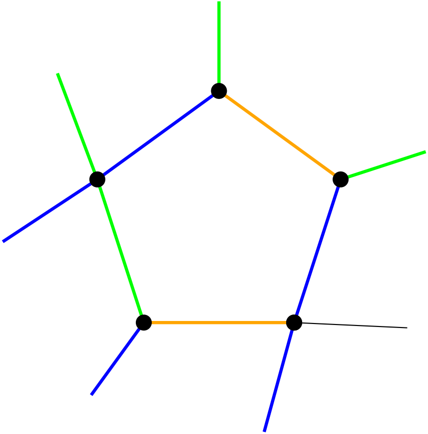

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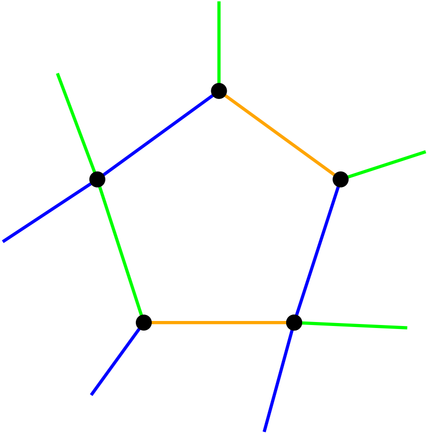

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- The medial graph $M(G)$ of a plane graph $G$:
  - $V(M(G)) = E(G)$;
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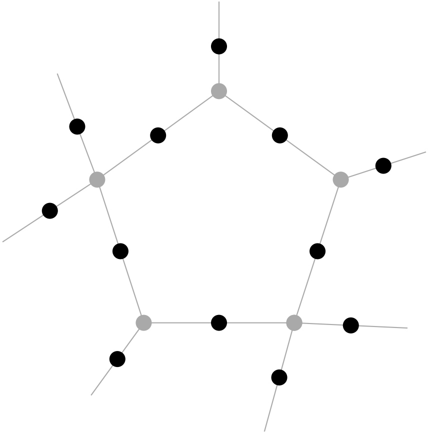

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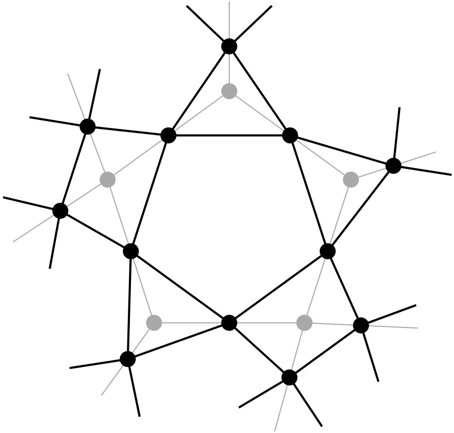

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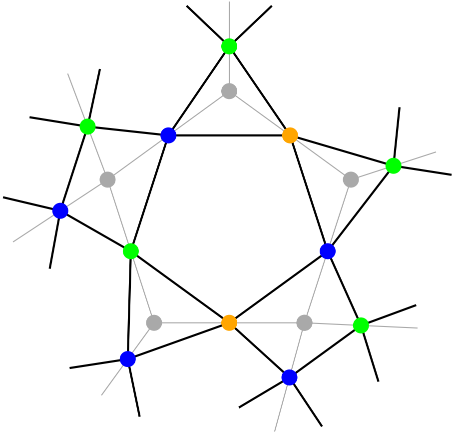

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- Deciding whether a planar graph $G$ with $\Delta(G) = 4$ admits a 3-coloring is NP-complete [7];
- $\rightarrow$ Lots of attention given to 3-colorability.

# 3-Colorability of Planar Graphs

**Theorem 2 (Heawood [10])**

*A plane triangulation is 3-colorable if and only if all its vertices have even degree.*

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- Improved by Grünbaum (and Aksenov) to planar graphs with at most three triangles.

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- Disproved by Cohen-Addad, Hebdige, Král', Li, and Salgado [2].

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- Open: Are planar graphs without cycles of lengths from 4 to 7 (or even 6) 3-choosable?

# Our Result

**Theorem 7 (Dross, BL, Maceková & Soták – 2018$^{+}$)**

*Every loopless planar graph with maximum degree 4 obtained as a subgraph of the medial graph of a bipartite plane graph is 3-choosable.*

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- Answer to Problem 1 also in the list setting.

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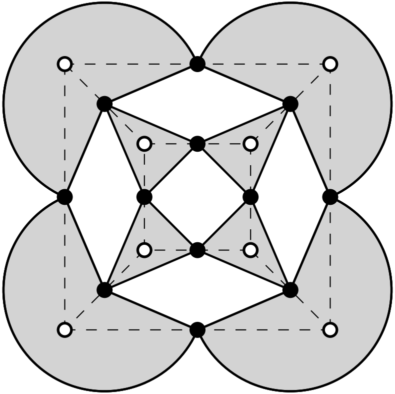
  - $\Rightarrow$ Every edge in $G$ is incident with one black and one white face;
  - Triangles are close & there are short cycles $\rightarrow$ still 3-choosable!

# Sketch of Proof – 2



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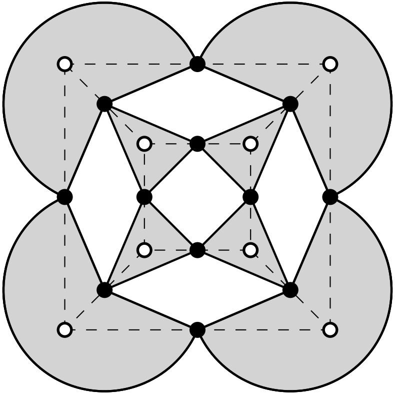

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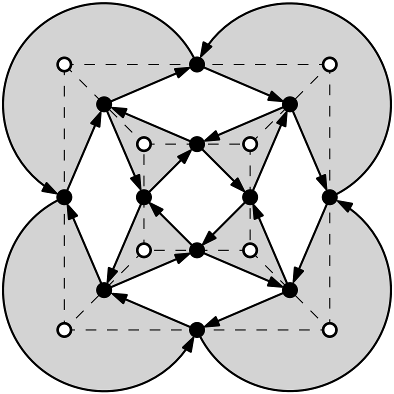

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*Let $D$ be a directed graph, and let $L$ be a list-assignment such that $|L(v)| \geq d_D^+(v) + 1$ for each $v \in V(D)$. If $E^e(D) \neq E^o(D)$, then $D$ is $L$-colorable.*

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- We need to prove that the number of even spanning Eulerian subgraphs is different from the number of odd spanning Eulerian subgraphs in $G$.

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  - The exterior $\mathrm{ext}(H)$ is the graph induced by the vertices of $G$ lying in the green faces of $H$ together with the vertices of $H$ without the edges of $H$.
- For a subgraph $X$ of $G$, we define:

$$\partial_X(H) = \partial(H) \cap X,\ \mathrm{int}_X(H) = \mathrm{int}(H) \cap X,\ \mathrm{ext}_X(H) = \mathrm{ext}(H) \cap X.$$

# Sketch of Proof – 5

## Observation 1

*Let $D_1$ and $D_2$ be two directed cycles in $G$ intersecting (i.e., having some common vertices) in such a way that $\partial(D_2) \cap \mathrm{int}(D_1) \neq \emptyset$ and $\partial(D_2) \cap \mathrm{ext}(D_1) \neq \emptyset$. Then $E(D_1) \cap E(D_2) \neq \emptyset$.*

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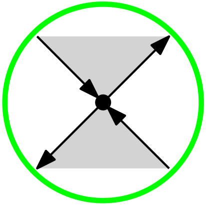

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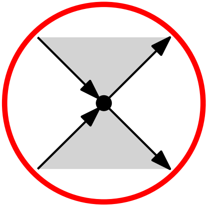

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- By Observation 1, all edges of a given directed cycle $C$ are incident either to black faces or to white faces in the interior of $C$;
- $\Rightarrow$ We distinguish two types of directed cycles in $G$:
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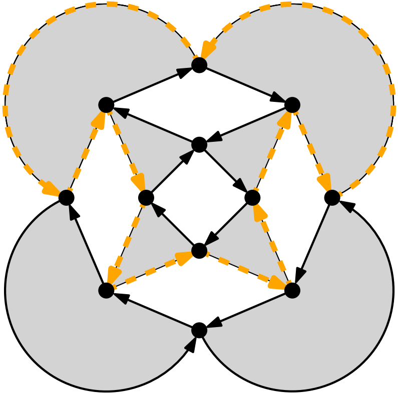

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- For a cycle $D$, the $D$-complement of a spanning Eulerian subgraph $X$ of $G$ is the spanning Eulerian subgraph $\overline{X}^D$ with the edge set

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- $\overline{X}^D$ is also Eulerian by Observation 1.

# Sketch of Proof – 8

**Claim 1**

*For an odd black cycle $D$, the $D$-complement of an odd (even) Eulerian spanning subgraph $X$ is an even (odd) Eulerian spanning subgraph $\overline{X}^D$.*

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*For an odd black cycle $D$, the $D$-complement of an odd (even) Eulerian spanning subgraph $X$ is an even (odd) Eulerian spanning subgraph $\overline{X}^D$.*

**Claim 2**

*Let $X$ be an Eulerian spanning subgraph of $G$, and let $D$ be a white odd Eulerian subgraph of $X$. Then, there is an odd black cycle in $\mathrm{int}_X(D)$ or $\mathrm{int}_{\overline{X}^D}(D)$.*

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- If in the step $i$ we remove from $\mathcal{E}$ some $X$, then we also remove its $C_i$-complement;
- Such pairs are always removed at the same step:

**Claim 3**

*The number of odd Eulerian spanning subgraphs removed from $\mathcal{E}$ at step $i$ is equal to the number of even such subgraphs.*

# Sketch of Proof – 10

- After all cycles from $\mathcal{O}$ are removed, there is no odd Eulerian spanning subgraph left in $\mathcal{E}$;

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- $\Rightarrow$ There is at least one even Eulerian spanning subgraph, containing at least one edge of every odd cycle in $G$, but not all edges of any!

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$\square$

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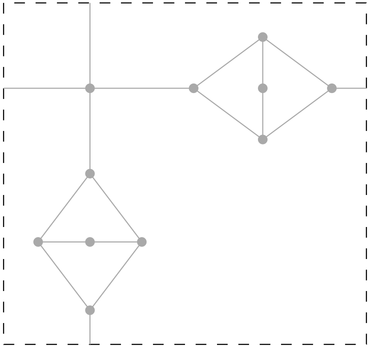

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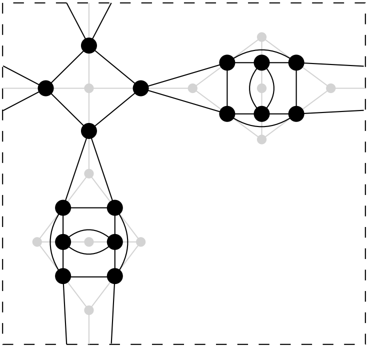

# Further Work

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**Conjecture 9**

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**Question 10**

*Is every simple plane graph whose faces can be properly colored with two colors such that one color class contains only even faces also 3-choosable?*

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Thank you!