# Star Edge-Coloring

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53rd Czech-Slovak Conference – Graphs 2018

May 30, 2018

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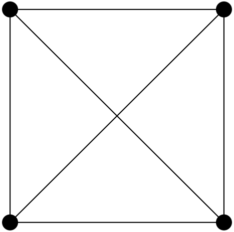
- Initiated by Liu and Deng in 2008 [8].

# Example



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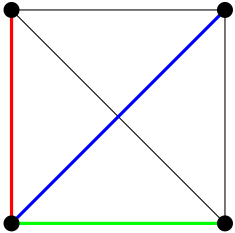

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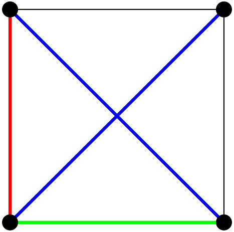

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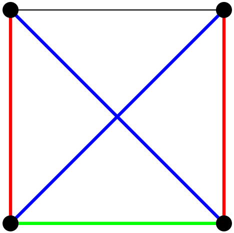

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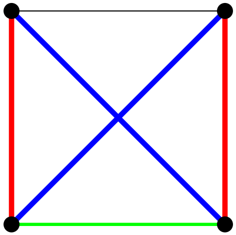

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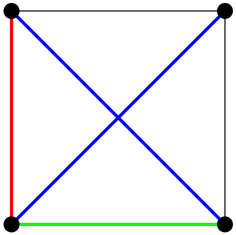

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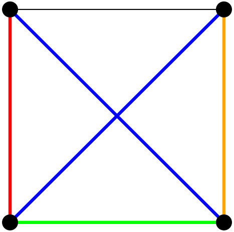

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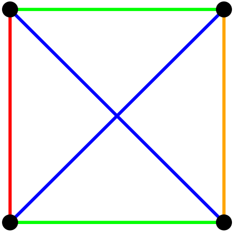

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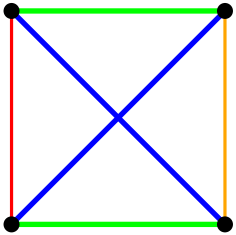

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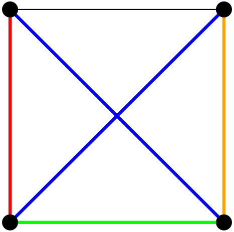

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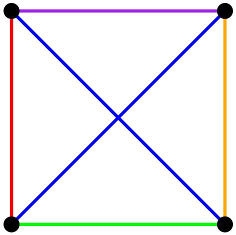

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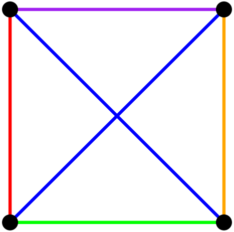

$$\chi'_{\mathrm{st}}(K_4) = 5$$

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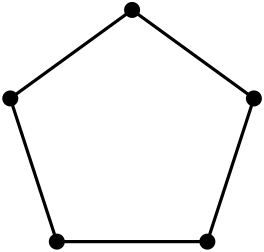

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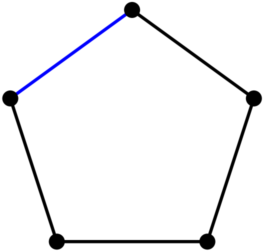

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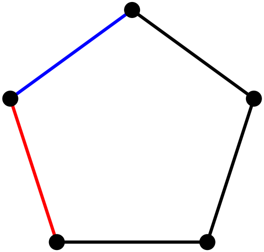

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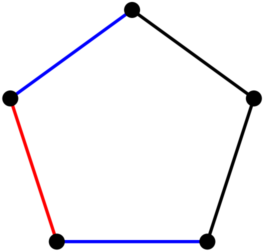

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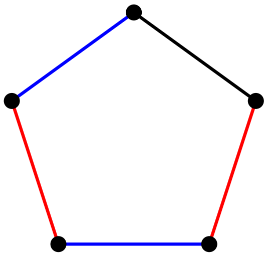

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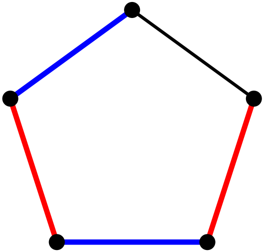

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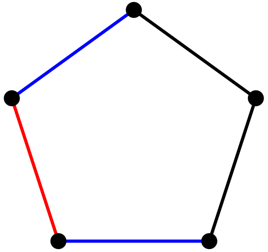

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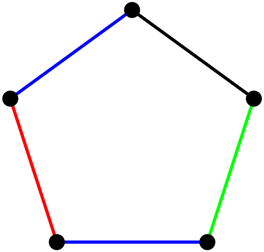

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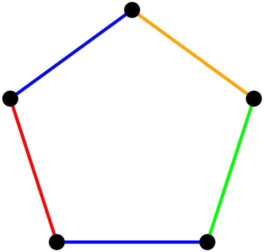
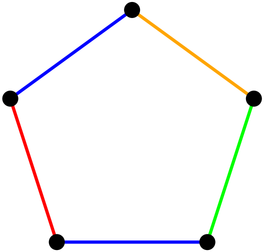


# Example

$$\chi'_{\mathrm{st}}(C_5) = 4$$



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**Observation 1 (Paths)**

*For any positive $n$, $\chi'_{\mathrm{st}}(P_n) = \min\{3, n-1\}$.*

**Observation 2 (Cycles)**

*For any positive $n \neq 5$, $\chi'_{\mathrm{st}}(C_n) = 3$; $\chi'_{\mathrm{st}}(C_5) = 4$.*

# Complete Graphs - Upper Bound

**Theorem 3 (Dvořák, Mohar, Šámal [2])**

*The star chromatic index of the complete graph $K_n$ satisfies*

$$\chi'_{\mathrm{st}}(K_n) \leq n \frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{1/4}} .$$

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- $\rightarrow$ using result on the size of a subset of $\{1,2,\ldots,N\}$ without 3-term arithmetic progression.

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- Exact equality (without ceiling) attained for $n \in \{1, 2, 8\}$, when every color appears same number of times;
- It cannot be true for $n \in \{6, 11, 16\}$, i.e. $\chi'_{\mathrm{st}}(K_6) \geq 10$, $\chi'_{\mathrm{st}}(K_{11}) \geq 23$, $\chi'_{\mathrm{st}}(K_{16}) \geq 37$;
- Not (yet) known for $n \in \{26, 56\}$.

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**Conjecture 5 (Dvořák, Mohar, Šámal [2])**

*The star chromatic index of the complete graph $K_n$ is linear in $n$, i.e.,*

$$\chi'_{\mathrm{st}}(K_n) \in \mathcal{O}(n).$$

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$$\{d(u,v) \mid uv \in M\} = \{1, 2, \ldots, \ell\},$$

where $u < v$ and $d(u,v) = \min\{v-u, (n-(v-u))\}$;

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- Color edges of $M$ with 3 colors:

$$\varpi \ : \ M \to \{A, B, C\}$$

## Rotational Coloring

- **Rotate** $M$ by $i$ (modulo $n$), $i \in \{0, 1, \ldots, n-1\}$:

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- If $\varphi$ is star edge-coloring, we call it **rotational star 3-edge-coloring**.

# Rotational Coloring

**Proposition 6 (BL, Mockovčiaková, Soták, $2014^{+}$)**

*For every odd $n$, $1 \leq n \leq 19$, there is a rotational star 3-edge-coloring of $K_n$. Moreover, for $n = 21$, such a coloring does not exist.*

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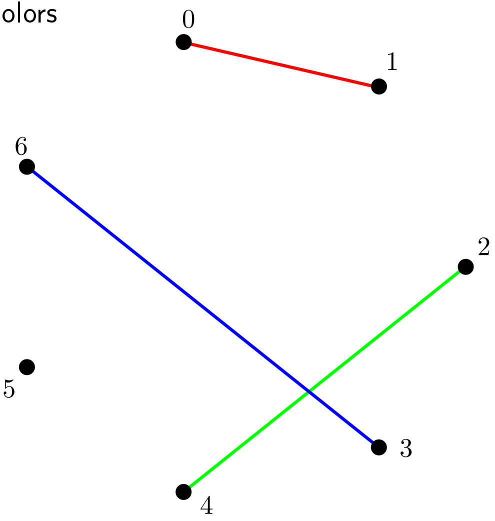
- **Proposition** 6 verified by computer;
- For $n = 23$ currently running;
  1515283 matchings satisfying the 'spanning' assumption;
  39366 possible (promising) colorings of a single matching.

# Rotational Coloring - Examples

- $K_7$: 21 colors



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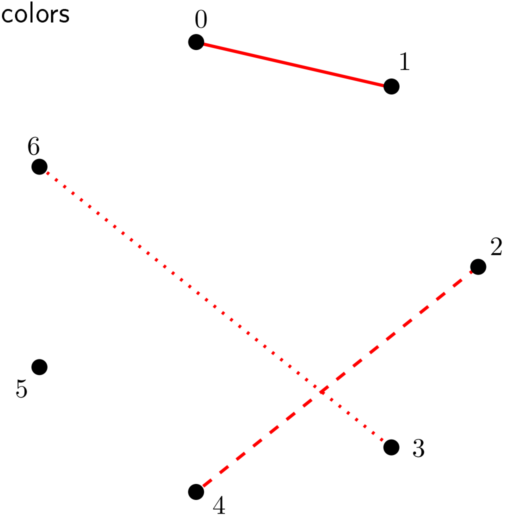

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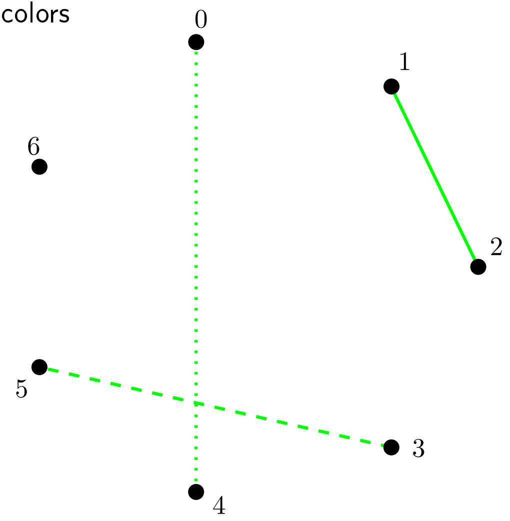

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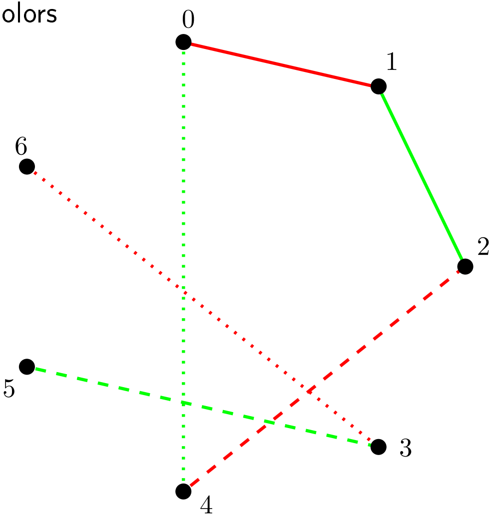

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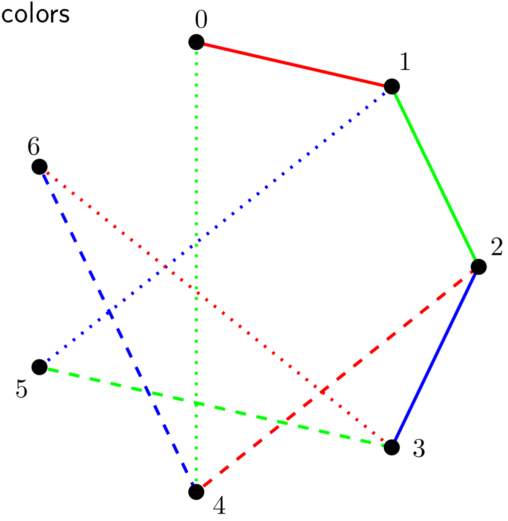

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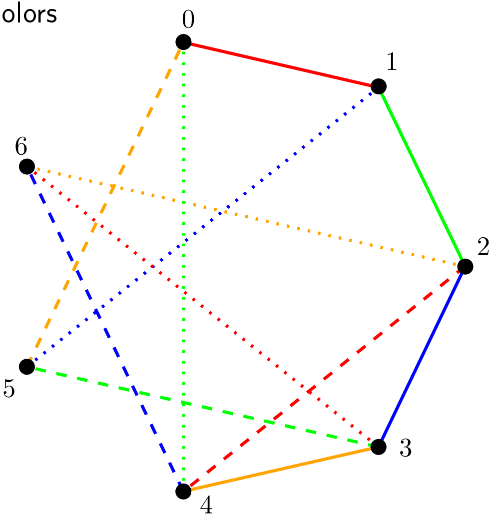

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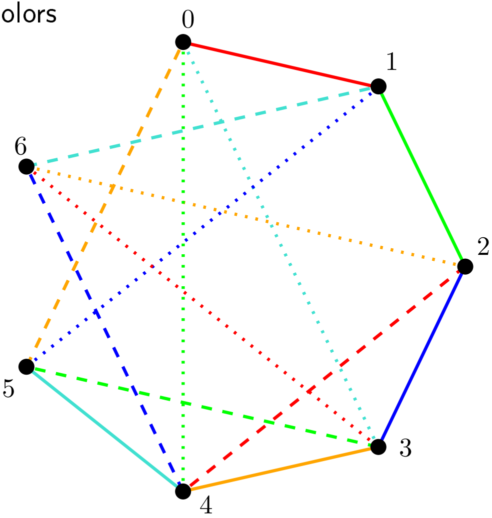

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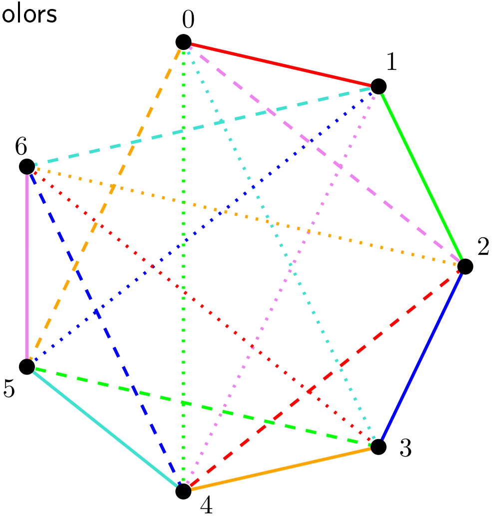

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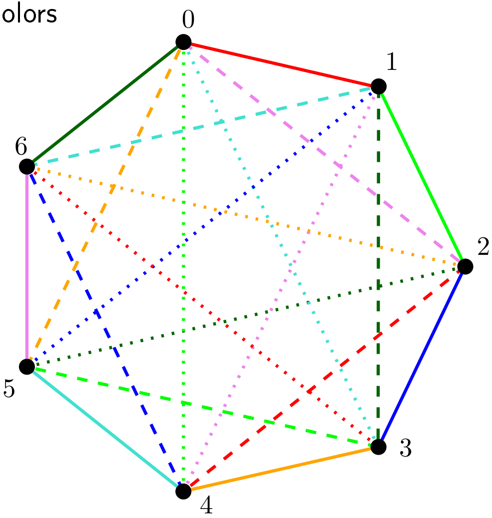

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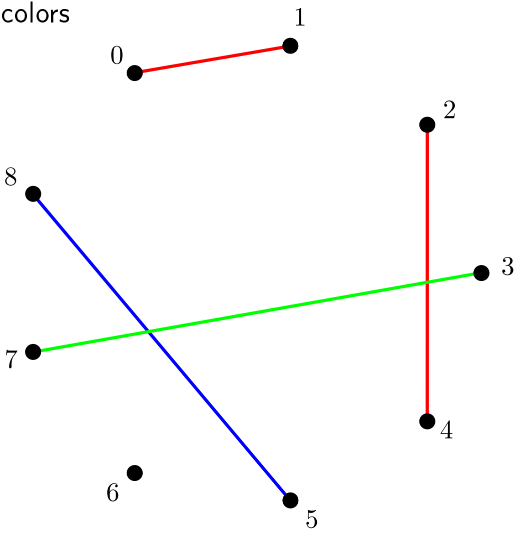
- $K_9$: 27 colors



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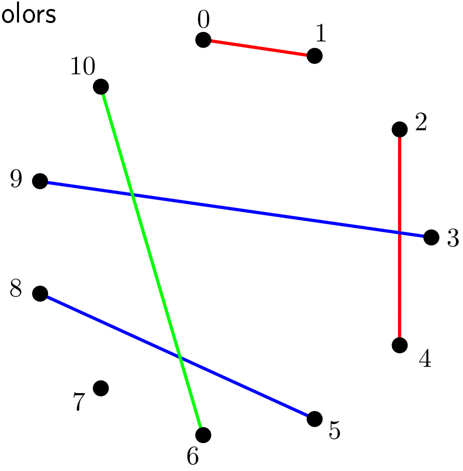
- $K_{11}$: 33 colors



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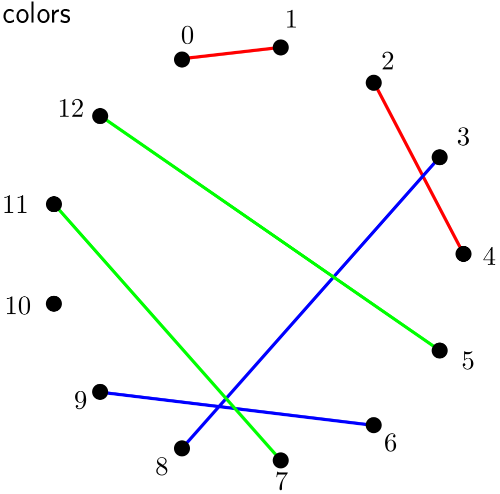
- $K_{13}$: 39 colors



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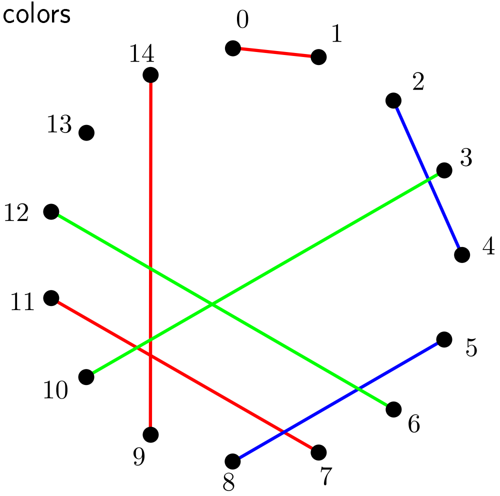
- $K_{15}$: 45 colors



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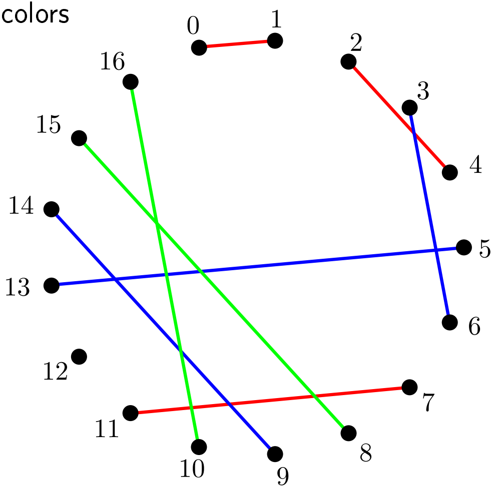
- $K_{17}$: 51 colors



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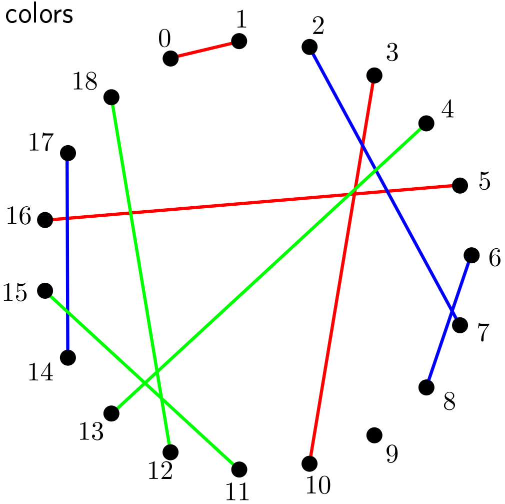
- $K_{19}$: 57 colors



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**Question 7**

*Does linear number of colors for star edge-coloring of $K_n$ imply linear number of colors for rotational star edge-coloring of $K_n$?*

**Question 8**

*Is Conjecture 5 somehow 'equivalent' to Perfect One Factorization Conjecture?*

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**Observation 9 (Dvořák, Mohar, Šámal [2])**

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**Observation 10 (Dvořák, Mohar, Šámal [2])**

$$\chi'_{\mathrm{st}}(K_n) \le \sum_{i=1}^{\lceil \log_2 n \rceil} 2^{i-1} \chi'_{\mathrm{st}}(K_{\lceil n/2^i \rceil, \lceil n/2^i \rceil}).$$

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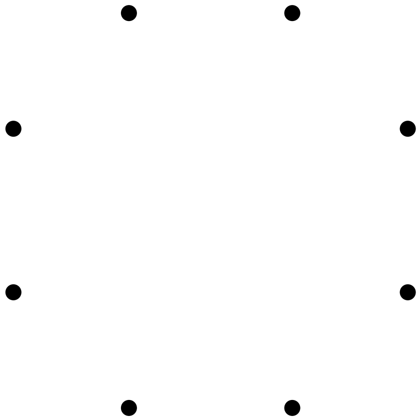
- **Sketch**:



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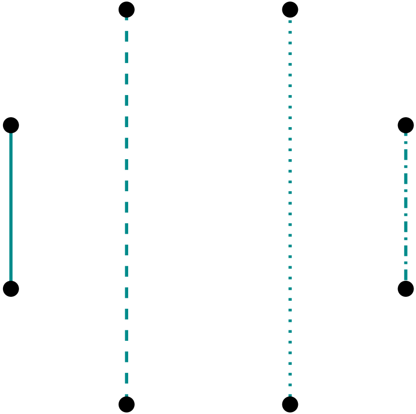

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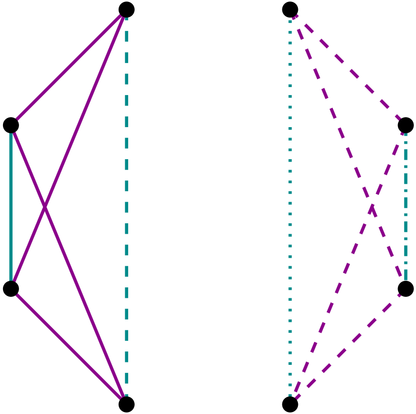

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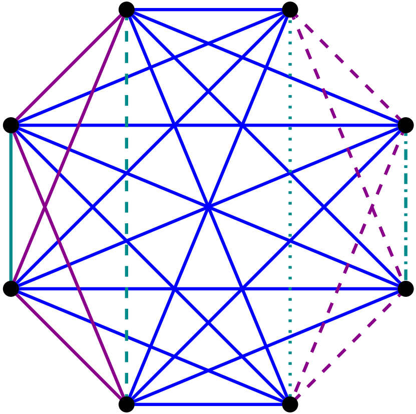

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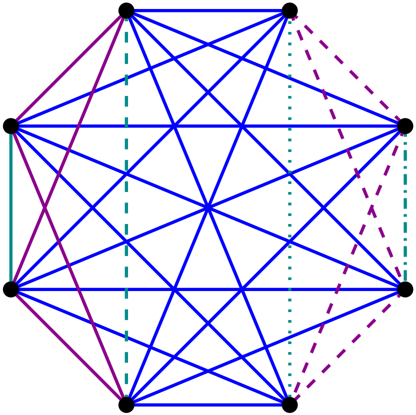

- No bichromatic component from two color bundles.

# Computer Assisted Bounds

| $n$ | $\chi_{\mathrm{st}}(K_{n,n})$ | $\chi_{\mathrm{st}}(K_n)$ A304525 |
|---|---|---|
| 1 | 1 | 0 |
| 2 | 3 | 1 |
| 3 | 6 | 3 |
| 4 | 7 | 5 |
| 5 | 11 | 9 |
| 6 | 13 | 12 |
| 7 | $13 \leq \cdot \leq 14$ | 14 |
| 8 | $15 \leq \cdot \leq 21$ | 14 |
| 9 | $17 \leq \cdot \leq 24$ | 18 |
| 10 | $19 \leq \cdot \leq 30$ | $20 \leq \cdot \leq 22$ |

# General Graphs

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**Theorem 11 (Dvořák, Mohar, Šámal [2])**

*For a graph $G$ it holds*

$$\chi'_{\mathrm{st}}(G) \leq \chi'_{\mathrm{st}}(K_{\Delta(G)+1}) \cdot O\left(\frac{\log \Delta(G)}{\log \log \Delta(G)}\right)^2,$$

*and therefore* $\chi'_{\mathrm{st}}(G) \leq \Delta(G) \cdot 2^{O(1)\sqrt{\log \Delta(G)}}$.

# To more sparse graphs...

# Trees and Outerplanar Graphs

**Theorem 12 (Bezegová et al. [1])**

*For a tree $T$ it holds*

$$\chi'_{\mathrm{st}}(T) \leq \left\lfloor \frac{3\Delta(T)}{2} \right\rfloor .$$

# Trees and Outerplanar Graphs

**Theorem 12 (Bezegová et al. [1])**

*For a tree $T$ it holds*

$$\chi'_{\mathrm{st}}(T) \leq \left\lfloor \frac{3\Delta(T)}{2} \right\rfloor .$$

- Using the above result and taking a BFS tree of an outerplanar graph:

**Theorem 13 (Bezegová et al. [1])**

*For an outerplanar graph $G$ it holds*

$$\chi'_{\mathrm{st}}(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor + 12 .$$

# Outerplanar Graphs

**Conjecture 14 (Bezegová et al. [1])**

*For an outerplanar graph $G$ it holds*

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- Recent result:

**Theorem 15 (Wang, Wang & Wang [11])**

*For an outerplanar graph $G$ it holds*

$$\chi'_{\mathrm{st}}(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor + 5\,.$$

# Theorem for Minor Closed Graphs

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**Theorem 16 (Wang, Wang & Wang [11])**

*Let $\{F, H\}$ be an edge-partition of a graph $G$. Then*

$$\chi'_{\mathrm{st}}(G) \leq \chi'_{\mathrm{st}}(F) + \chi'_s(H|_G).$$

- Result for **strong** edge-coloring:

**Theorem 17 (Faudree et al. [3])**

*For a planar graph $G$ it holds*

$$\chi'_s(G) \leq 4\chi'(G).$$

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**Proof**:

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- Such **edges at distance** 2 have **different colors in** $G/M_i$;
- Coloring $e \in E(G)$ with $(\varphi(e), \tau_i(e))$ gives **strong edge-coloring** with at most $\chi'(G) \cdot 4$ **colors**.

# Planar Graphs

**Theorem 18 (Wang, Hu & Wang [10])**

*Every planar graph $G$ has an edge-decomposition into two forests $F_1$, $F_2$ and a subgraph $K$ such that $\Delta(K) \leq 10$ and $\Delta(F_i) \leq \lceil (\Delta(G) - 9)/2 \rceil$ for $i \in \{1, 2\}$.*

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- Using above and Theorem 16, currently the best bound for planar graphs can be obtained.

**Theorem 19 (Wang, Wang & Wang [11])**

*Let $G$ be a planar graph. Then*

$$\chi'_{\mathrm{st}}(G) \leq 2.75\Delta(G) + 18;$$

# Planar Graphs

- Similarly they proved more specific results (together with the result for outerplanar graphs from Theorem 15)

**Theorem 20 (Wang, Wang & Wang [11])**

*Let $G$ be a planar graph. Then*

(a) $\chi'_{\mathrm{st}}(G) \leq 2.25\Delta(G) + 6$, *if $G$ is $K_4$-minor free;*

(b) $\chi'_{\mathrm{st}}(G) \leq 1.5\Delta(G) + 18$, *if $G$ has no 4-cycles;*

(c) $\chi'_{\mathrm{st}}(G) \leq 1.5\Delta(G) + 13$, *if $G$ has girth at least 5;*

(d) $\chi'_{\mathrm{st}}(G) \leq 1.5\Delta(G) + 3$, *if $G$ has girth at least 8.*

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- $\mathrm{ch}'_{\mathrm{st}}(G)$: the **list star chromatic index** of $G$;

**Theorem 21**

*Let $G$ be a graph. Then*

(a) $\mathrm{ch}'_{\mathrm{st}}(G) \leq 2\Delta(G) - 1$ *if* $\mathrm{mad}(G) < 7/3$ *[6];*

(b) $\mathrm{ch}'_{\mathrm{st}}(G) \leq 2\Delta(G)$ *if* $\mathrm{mad}(G) < 5/2$ *[6];*

(c) $\mathrm{ch}'_{\mathrm{st}}(G) \leq 2\Delta(G) + 1$ *if* $\mathrm{mad}(G) < 8/3$ *[6];*

(d) $\mathrm{ch}'_{\mathrm{st}}(G) \leq 2\Delta(G) + 2$ *if* $\mathrm{mad}(G) < 14/5$ *[5];*

(e) $\mathrm{ch}'_{\mathrm{st}}(G) \leq 2\Delta(G) + 3$ *if* $\mathrm{mad}(G) < 3$ *[5];*

# Planar Graphs

- In [5] and [6] the authors are asking: Is there a constant $C$ such that for any planar graph $G$ $\chi'_{\mathrm{st}}(G) \leq 2\Delta(G) + C$;

# Planar Graphs

- In [5] and [6] the authors are asking: Is there a constant $C$ such that for any planar graph $G$ $\chi'_{\mathrm{st}}(G) \leq 2\Delta(G) + C$;
- We are not aware of any example needing $2\Delta$ colors, in fact, we believe even the question below has an affirmative answer:

**Question 22**

*Is there a constant $C$ such that for any planar graph $G$ it holds*

$$\chi'_{\mathrm{st}}(G) \leq \frac{3}{2}\Delta(G) + C.$$

# ... and very sparse graphs

# Subcubic Graphs

- The most analyzed class are subcubic graphs

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**Theorem 23 (Dvořák, Mohar, Šámal [2])**

(a) *If $G$ is a subcubic graph, then $\chi'_{\mathrm{st}}(G) \leq 7$.*

(b) *If $G$ is a simple cubic graph, then $\chi'_{\mathrm{st}}(G) \geq 4$, and the equality holds if and only if $G$ covers the graph of the 3-cube.*

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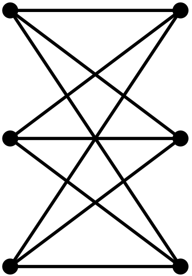
**Conjecture 24 (Dvořák, Mohar, Šámal [2])**

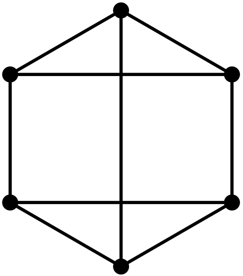
*If $G$ is a subcubic graph, then $\chi'_{\mathrm{st}}(G) \leq 6$.*

# Subcubic Graphs

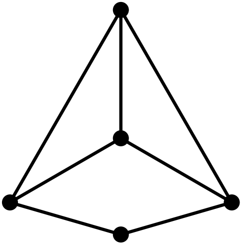
- Only **three** known **2-connected graphs** needing **6 colors**:



$K_{3,3}$



$\overline{C_6}$



$K_4 + v$

# Subcubic Graphs

- A number of partial results:

**Theorem 25**

*Let $G$ be a graph with maximum degree 3. Then*

(a) $\chi'_{\mathrm{st}}(G) \leq 5$ *if $G$ is outerplanar [1];*

(b) $\chi'_{\mathrm{st}}(G) \leq 5$ *if* $\mathrm{mad}(G) < \frac{12}{5}$ *[7];*

(c) $\chi'_{\mathrm{st}}(G) \leq 5$ *if* $\mathrm{mad}(G) < \frac{7}{3}$ *(in the list setting!) [4];*

(d) $\chi'_{\mathrm{st}}(G) \leq 6$ *if* $\mathrm{mad}(G) < \frac{5}{2}$ *(in the list setting!) [4].*

# Subcubic Graphs - List Version

**Question 26 (Dvořák, Mohar, Šámal [2])**

*Is it true that* $\mathrm{ch}'_{\mathrm{st}}(G) \leq 7$ *for every subcubic graph* $G$*? (Perhaps even* $\leq 6$*?)*

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**Theorem 27 (BL, Mockovčiaková & Soták [9])**

*For every subcubic graph* $G$*, it holds*

$$\mathrm{ch}'_{\mathrm{st}}(G) \leq 7.$$

# Another nice class

## Hypercubes

- In $Q_n$, the edges **in every dimension $i$ can be divided** in two sets, $A_i$ and $B_i$ such that the edges in each set are **at distance at least $3$**;

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**Conjecture 28**

*There is a constant $C$ such that for every positive $n$*

$$\chi'_{\mathrm{st}}(Q_n) = 2n - C\log(n).$$

# Further open problems

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**Question 29 (Dvořák, Mohar, Šámal, 2013)**

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- Is above true at least for **bipartite** ones? Or the ones with **large girth**?

Děkuji / Ďakujem!

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