Star Edge-Coloring

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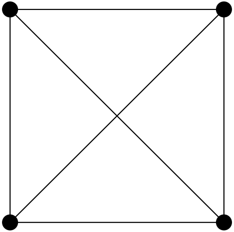
- Initiated by Liu and Deng in 2008 [8].

Example



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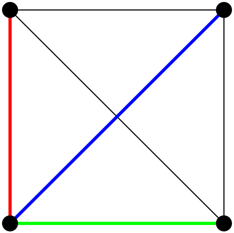

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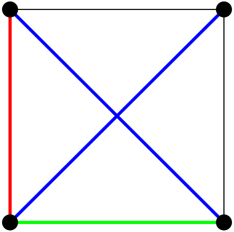

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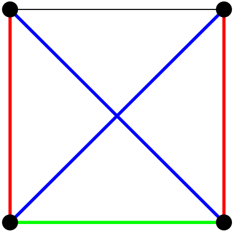

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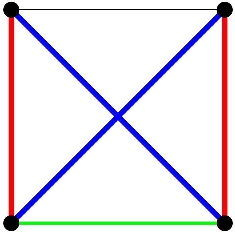

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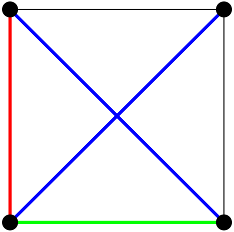

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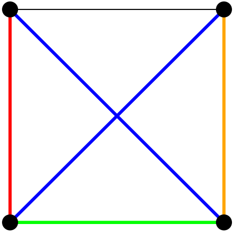

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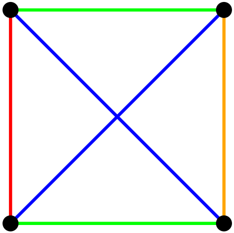

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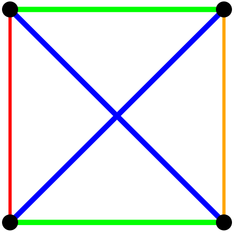

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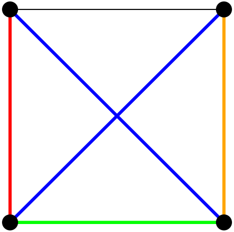

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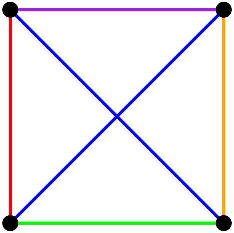

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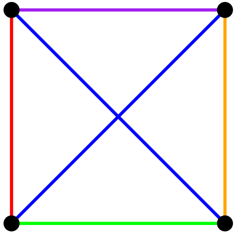
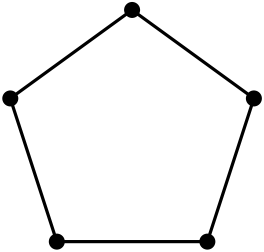


$\chi'_{\mathrm{st}}(K_4) = 5$

Example



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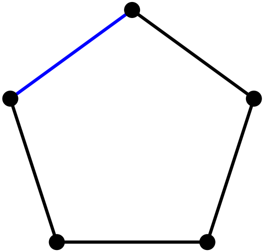

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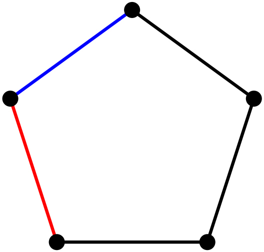

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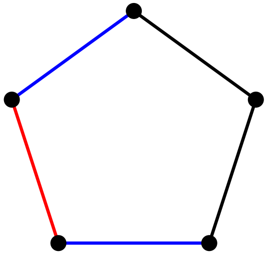

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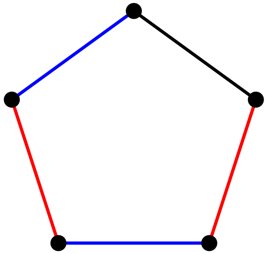

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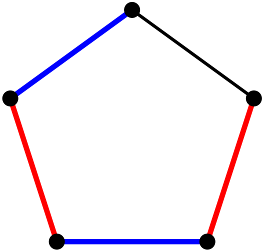

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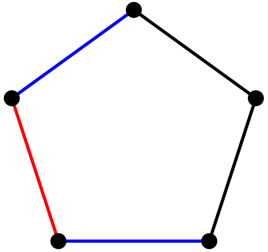

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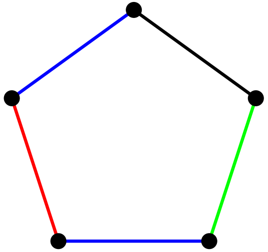

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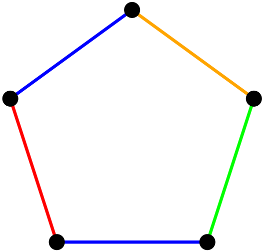
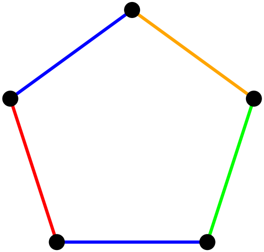


Example

$\chi'_{\mathrm{st}}(C_5) = 4$



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For any positive n, $\chi'_{\mathrm{st}}(P_n) = \min\{3, n-1\}$.

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Observation 1 (Paths)

For any positive n, $\chi'_{\mathrm{st}}(P_n) = \min\{3, n-1\}$.

Observation 2 (Cycles)

For any positive $n \neq 5$, $\chi'_{\mathrm{st}}(C_n) = 3$; $\chi'_{\mathrm{st}}(C_5) = 4$.

Complete Graphs - Upper Bound

Theorem 3 (Dvořák, Mohar, Šámal [2])

The star chromatic index of the complete graph K_n satisfies

$$\chi'_{\mathrm{st}}(K_n) \le n \frac{2^{2\sqrt{2}(1+o(1))\sqrt{\log n}}}{(\log n)^{1/4}}.$$

In particular, for every $\epsilon > 0$ there exists a constant C such that $\chi'_{\mathrm{st}}(K_n) \le C\, n^{1+\epsilon}$ for every $n \ge 1$.

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- $\rightarrow$ using result on the size of a subset of {1,2,...,N} without 3-term arithmetic progression.

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- Exact equality (without ceiling) attained for $n \in \{1, 2, 8\}$, when every color appears same number of times;
- It cannot be true for $n \in \{6, 11, 16\}$, i.e. $\chi'_{\mathrm{st}}(K_6) \geq 10$, $\chi'_{\mathrm{st}}(K_{11}) \geq 23$, $\chi'_{\mathrm{st}}(K_{16}) \geq 37$;
- Not (yet) known for $n \in \{26, 56\}$.

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Conjecture 5 (Dvořák, Mohar, Šámal [2])

The star chromatic index of the complete graph K_n is linear in n, i.e.,

$$\chi'_{\mathrm{st}}(K_n) \in \mathcal{O}(n).$$

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- Color edges of M with 3 colors:

$$\pi \ : \ M \to \{A, B, C\}$$

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- **Rotate** M by i (modulo n), $i \in \{0, 1, \ldots, n-1\}$:

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- If φ is star edge-coloring, we call it **rotational star 3-edge-coloring**.

Rotational Coloring

Proposition 6 (BL, Mockovčiaková, Soták, 2014^+)

For every odd n, $1 \leq n \leq 19$, there is a rotational star 3-edge-coloring of K_n. Moreover, for $n = 21$, such a coloring does not exist.

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- **Proposition** 6 verified by computer;

Rotational Coloring - Examples

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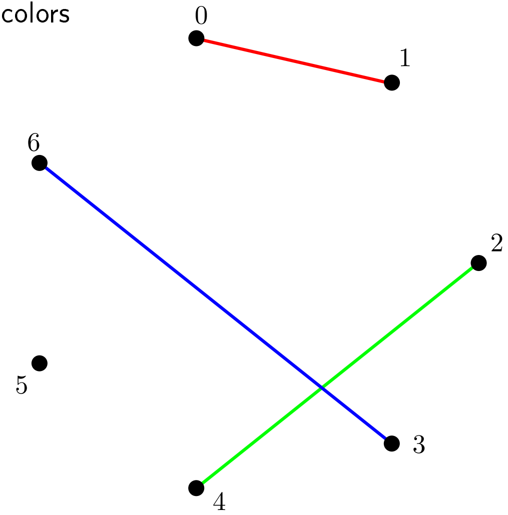

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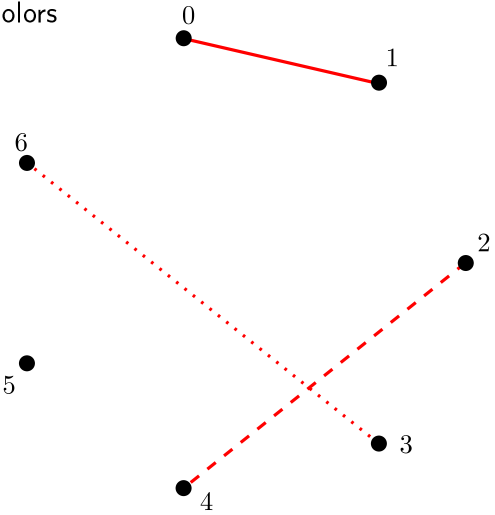

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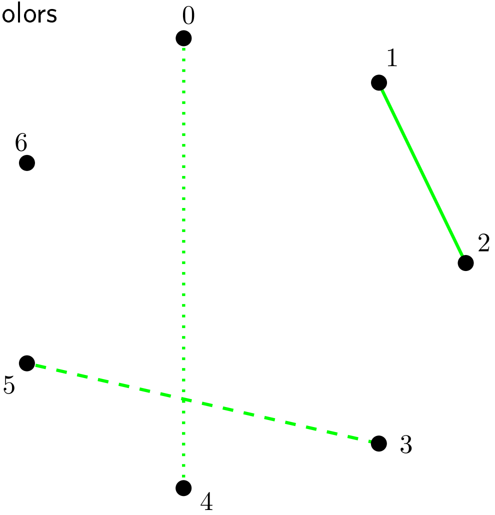

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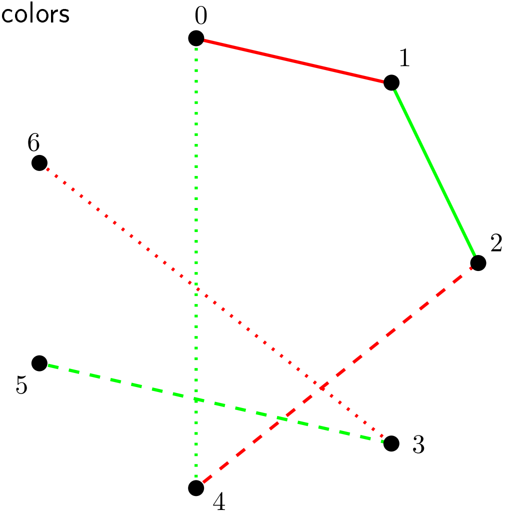

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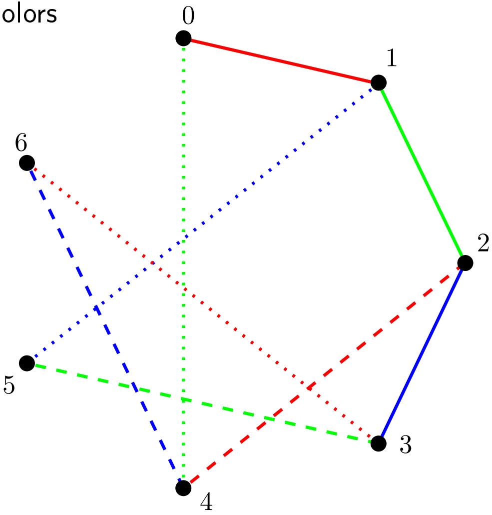

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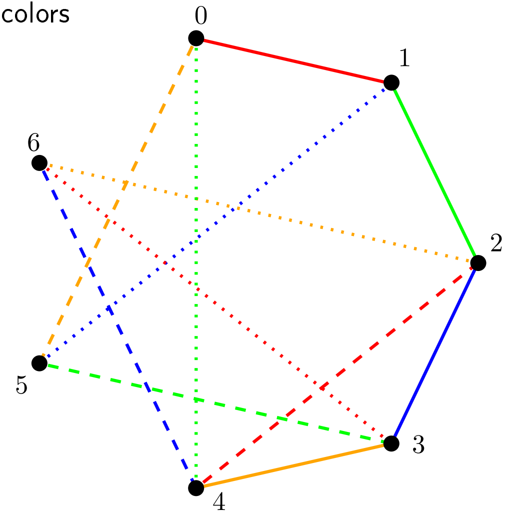

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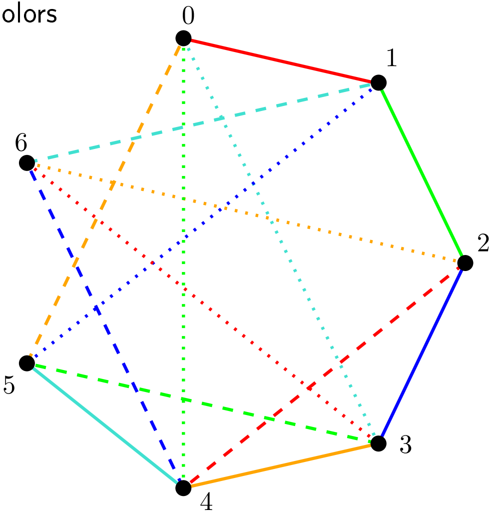

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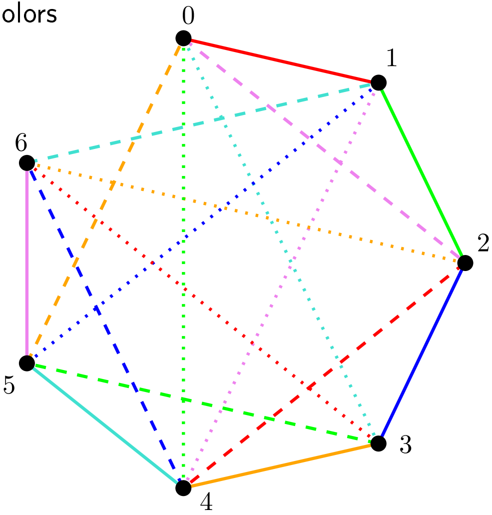

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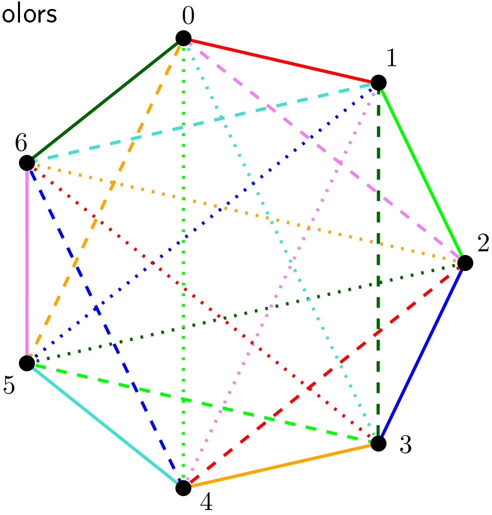

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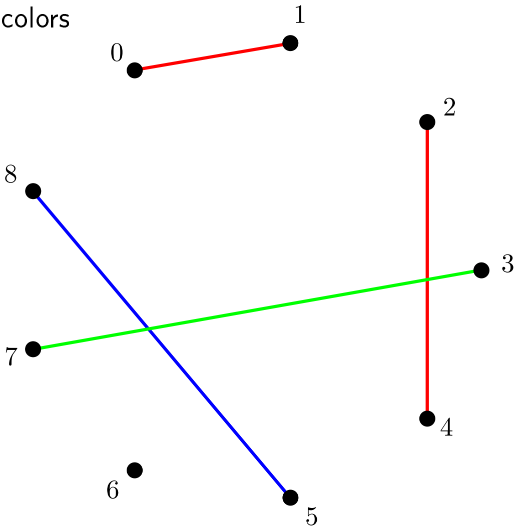
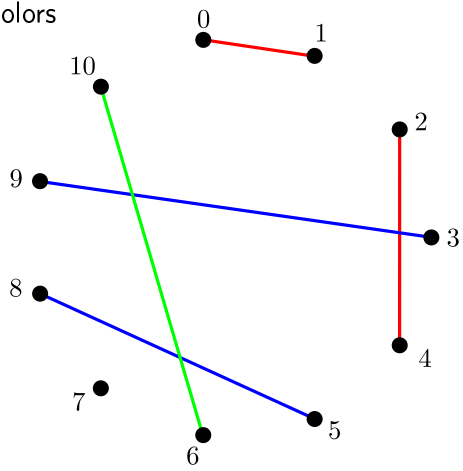
- K_9: 27 colors



Rotational Coloring - Examples

- K_{11}: 33 colors



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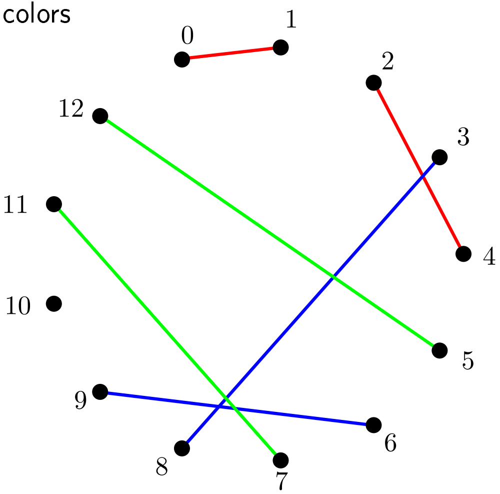
- K_{13}: 39 colors



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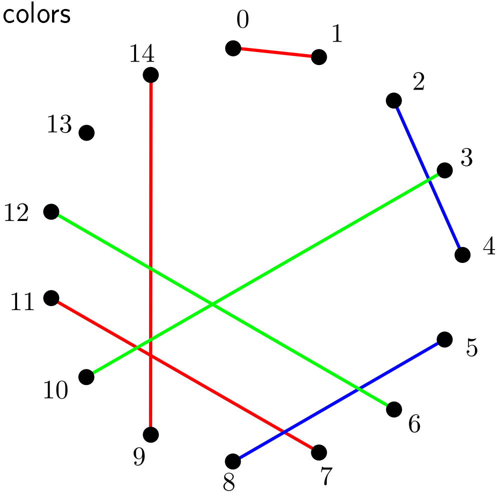
- K_{15}: 45 colors



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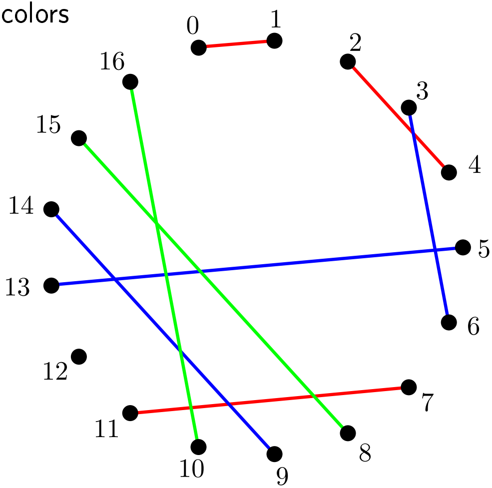
- K_{17}: 51 colors



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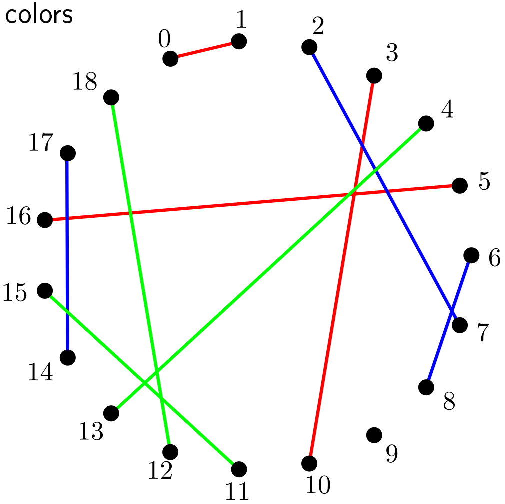
- K_{19}: 57 colors



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Question 7

Does linear number of colors for star edge-coloring of K_n imply linear number of colors for rotational star edge-coloring of K_n?

Question 8

Is Conjecture 5 somehow 'equivalent' to Perfect One Factorization Conjecture?

Complete Bipartite Graphs

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Observation 10 (Dvořák, Mohar, Šámal [2])

$$\chi'_{\mathrm{st}}(K_n) \leq \sum_{i=1}^{\lceil \log_2 n \rceil} 2^{i-1} \chi'_{\mathrm{st}}(K_{\lceil n/2^i \rceil, \lceil n/2^i \rceil}).$$

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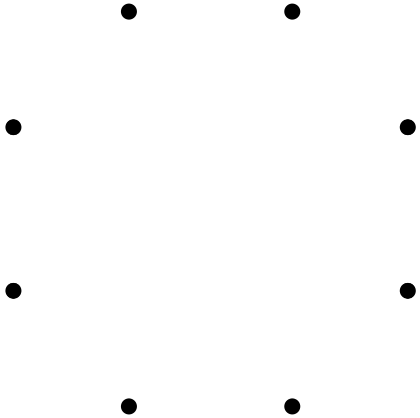
- **Sketch**:



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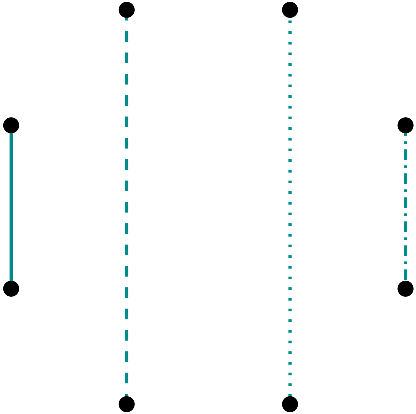

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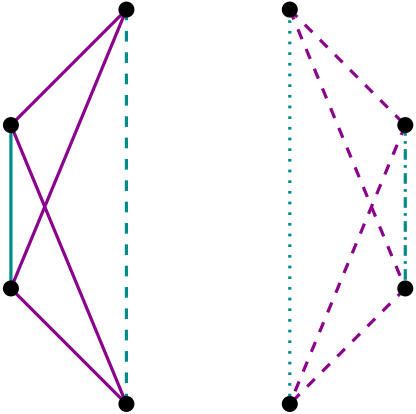

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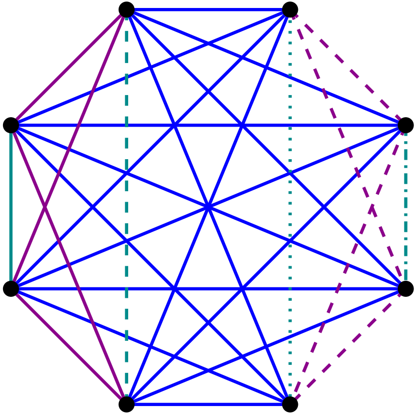

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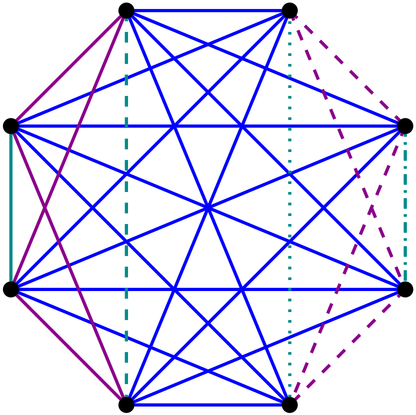

- No bichromatic component from two color bundles.

Computer Assisted Bounds

n	$\chi_{\mathrm{st}}(K_{n,n})$	$\chi_{\mathrm{st}}(K_n)$ A304525
1	1	0
2	3	1
3	6	3
4	7	5
5	11	9
6	13	12
7	14	14
8	15	14
9	$18 \leq \cdot \leq 24$	18
10	$19 \leq \cdot \leq 30$	$20 \leq \cdot \leq 22$

General Graphs

- Upper bound for general graphs is obtained **from the bound for complete graphs**;

General Graphs

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Theorem 11 (Dvořák, Mohar, Šámal [2])

For a graph G it holds

$$\chi'_{\mathrm{st}}(G) \le \chi'_{\mathrm{st}}(K_{\Delta(G)+1}) \cdot O\left(\frac{\log \Delta(G)}{\log\log \Delta(G)}\right)^2,$$

and therefore $\chi'_{\mathrm{st}}(G) \le \Delta(G) \cdot 2^{O(1)\sqrt{\log \Delta(G)}}$.

To more sparse graphs...

Trees and Outerplanar Graphs

Theorem 12 (Bezegová et al. [1])

For a tree T it holds

$$\chi'_{\mathrm{st}}(T) \leq \left\lfloor \frac{3\Delta(T)}{2} \right\rfloor .$$

Trees and Outerplanar Graphs

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For a tree T it holds

$$\chi'_{\mathrm{st}}(T) \leq \left\lfloor \frac{3\Delta(T)}{2} \right\rfloor .$$

- Using the above result and taking a BFS tree of an outerplanar graph:

Theorem 13 (Bezegová et al. [1])

For an outerplanar graph G it holds

$$\chi'_{\mathrm{st}}(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor + 12 .$$

Outerplanar Graphs

Conjecture 14 (Bezegová et al. [1])

For an outerplanar graph G it holds

$$\chi'_{\mathrm{st}}(G) \leq \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor + 1\,.$$

Outerplanar Graphs

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- Recent result:

Theorem 15 (Wang, Wang & Wang [11])

For an outerplanar graph G it holds

$$\chi'_{\mathrm{st}}(G) \le \left\lfloor \frac{3\Delta(G)}{2} \right\rfloor + 5\,.$$

Theorem for Minor Closed Graphs

- **Strong edge-coloring** of a graph G: proper edge-coloring where every three consecutive edges receive different colors;

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Theorem 16 (Wang, Wang & Wang [11])

Let $\{F, H\}$ be an edge-partition of a graph G. Then

$$\chi'_{\mathrm{st}}(G) \leq \chi'_{\mathrm{st}}(F) + \chi'_s(H|_G).$$

Planar Graphs

- Result for **strong** edge-coloring:

Theorem 17 (Faudree et al. [3])

For a planar graph G it holds

$$\chi'_s(G) \leq 4\chi'(G).$$

Proof:

- Color edges of G **properly**: ($\chi'(G)$ **colors**) $\rightarrow$ coloring φ;

Planar Graphs

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- Color edges of G **properly**: ($\chi'(G)$ **colors**) $\rightarrow$ coloring φ;
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Planar Graphs

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Planar Graphs

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- G/M_i is planar $\rightarrow$ 4-**vertex-colorable** $\rightarrow$ coloring τ_i;
- Vertices in G/M_i correspond to edges colored by i;
- Such **edges at distance** 2 have **different colors in** G/M_i;
- Coloring $e \in E(G)$ with $(\varphi(e), \tau_i(e))$ gives **strong edge-coloring** with at most $\chi'(G) \cdot 4$ **colors**.

Planar Graphs

Theorem 18 (Wang, Hu & Wang [10])

Every planar graph G has an edge-decomposition into two forests F_1, F_2 and a subgraph K such that $\Delta(K) \leq 10$ and $\Delta(F_i) \leq \lceil(\Delta(G) - 9)/2\rceil$ for $i \in \{1, 2\}$.

Planar Graphs

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- Using above and Theorem 16, currently the best bound for planar graphs can be obtained.

Theorem 19 (Wang, Wang & Wang [11])

Let G be a planar graph. Then

$$\chi'_{\mathrm{st}}(G) \leq 2.75\Delta(G) + 18;$$

Planar Graphs

- Similarly they proved more specific results (together with the result for outerplanar graphs from Theorem 15)

Theorem 20 (Wang, Wang & Wang [11])

Let G be a planar graph. Then

(a) $\chi'_{\mathrm{st}}(G) \leq 2.25\Delta(G) + 6$, *if G is K_4-minor free;*

(b) $\chi'_{\mathrm{st}}(G) \leq 1.5\Delta(G) + 18$, *if G has no 4-cycles;*

(c) $\chi'_{\mathrm{st}}(G) \leq 1.5\Delta(G) + 13$, *if G has girth at least 5;*

(d) $\chi'_{\mathrm{st}}(G) \leq 1.5\Delta(G) + 3$, *if G has girth at least 8.*

Graphs with Bounded mad

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- $\mathrm{ch}'_{\mathrm{st}}(G)$: the **list star chromatic index** of G;

Theorem 21

Let G be a graph. Then

(a) $\mathrm{ch}'_{\mathrm{st}}(G) \leq 2\Delta(G) - 1$ *if* $\mathrm{mad}(G) < 7/3$ *[6];*

(b) $\mathrm{ch}'_{\mathrm{st}}(G) \leq 2\Delta(G)$ *if* $\mathrm{mad}(G) < 5/2$ *[6];*

(c) $\mathrm{ch}'_{\mathrm{st}}(G) \leq 2\Delta(G) + 1$ *if* $\mathrm{mad}(G) < 8/3$ *[6];*

(d) $\mathrm{ch}'_{\mathrm{st}}(G) \leq 2\Delta(G) + 2$ *if* $\mathrm{mad}(G) < 14/5$ *[5];*

(e) $\mathrm{ch}'_{\mathrm{st}}(G) \leq 2\Delta(G) + 3$ *if* $\mathrm{mad}(G) < 3$ *[5];*

Planar Graphs

- In [5] and [6] the authors are asking: Is there a constant C such that for any planar graph G $\chi'_{\mathrm{st}}(G) \leq 2\Delta(G) + C$;

Planar Graphs

- In [5] and [6] the authors are asking: Is there a constant C such that for any planar graph G $\chi'_{\mathrm{st}}(G) \leq 2\Delta(G) + C$;
- We are not aware of any example needing 2Δ colors, in fact, we believe even the question below has an affirmative answer:

Question 22

Is there a constant C such that for any planar graph G it holds

$$\chi'_{\mathrm{st}}(G) \leq \frac{3}{2}\Delta(G) + C.$$

… and very sparse graphs

Subcubic Graphs

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Theorem 23 (Dvořák, Mohar, Šámal [2])

(a) *If G is a subcubic graph, then $\chi'_{\mathrm{st}}(G) \leq 7$.*

(b) *If G is a simple cubic graph, then $\chi'_{\mathrm{st}}(G) \geq 4$, and the equality holds if and only if G covers the graph of the 3-cube.*

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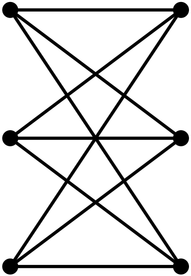
Conjecture 24 (Dvořák, Mohar, Šámal [2])

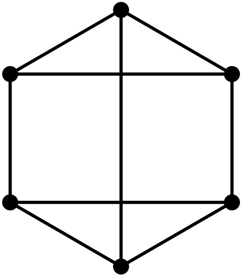
If G is a subcubic graph, then $\chi'_{\mathrm{st}}(G) \leq 6$.

Subcubic Graphs

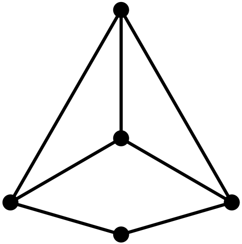
- Only **three** known 2-**connected graphs** needing 6 **colors**:



$K_{3,3}$



$\overline{C_6}$



$K_4 + v$

Subcubic Graphs

- A number of partial results:

Theorem 25

Let G be a graph with maximum degree 3. Then

(a) $\chi'_{\mathrm{st}}(G) \leq 5$ *if G is outerplanar [1];*

(b) $\chi'_{\mathrm{st}}(G) \leq 5$ *if* $\mathrm{mad}(G) < \frac{12}{5}$ *[7];*

(c) $\chi'_{\mathrm{st}}(G) \leq 5$ *if* $\mathrm{mad}(G) < \frac{7}{3}$ *(in the list setting!) [4];*

(d) $\chi'_{\mathrm{st}}(G) \leq 6$ *if* $\mathrm{mad}(G) < \frac{5}{2}$ *(in the list setting!) [4].*

Subcubic Graphs - List Version

Question 26 (Dvořák, Mohar, Šámal [2])

Is it true that $\mathrm{ch}'_{\mathrm{st}}(G) \leq 7$ *for every subcubic graph* G*? (Perhaps even* ≤ 6*?)*

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Is it true that $\mathrm{ch}'_{\mathrm{st}}(G) \leq 7$ *for every subcubic graph* G*? (Perhaps even* ≤ 6*?)*

Theorem 27 (BL, Mockovčiaková & Soták [9])

For every subcubic graph G*, it holds*

$$\mathrm{ch}'_{\mathrm{st}}(G) \leq 7.$$

Another nice class

Hypercubes

- In Q_n, the edges **in every dimension i can be divided** in two sets, A_i and B_i such that the edges in each set are **at distance at least** 3;

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Conjecture 28

There is a constant C such that for every positive n

$$\chi'_{\mathrm{st}}(Q_n) = 2n - C\log(n).$$

Further open problems

Open Problems

Question 29 (Dvořák, Mohar, Šámal, 2013)

Is it true that $\mathrm{ch}'_{\mathrm{st}}(G) = \chi'_{\mathrm{st}}(G)$ *for every graph* G*?*

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- Is it true that $\chi'_{\mathrm{st}}(G) \leq 5$ **for all 2-connected subcubic graphs** except a finite number of exceptions?
- Is above true at least for **bipartite** ones? Or the ones with **large girth**?

Merci!

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